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Stability of impulsive time-varying systems and compactness of the operators mapping the input space into the state and output spaces

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Abstract

This paper is concerned with time-varying systems with non-necessarily bounded everywhere continuous time-differentiable time-varying point delays. The delay-free and delayed dynamics are assumed to be time-varying and impulsive, in general, and the external input may also be impulsive. For given bounded initial conditions, the (unique) homogeneous state-trajectory and output trajectory are equivalently constructed from three different auxiliary homogeneous systems, the first one being delay-free and time-invariant, the second one possessing the delay-free dynamics of the current delayed system and the third one being the homogeneous part of the system under study. In this way, the constructed solution trajectories of both the unforced and forced systems are obtained from different (input-state space/output space and state space to output space) operators. The system stability and the compactness of the operators describing the solution trajectories are investigated.

Keywords: Compact operators; Time-delay dynamic systems; Stability; Time-varying dynamic systems

1. Introduction

Time-delay systems have been widely investigated in the last years both in a theoretical context and in that of related applications (see, for instance, [1–12]). Those systems become inher-

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ently attractive from a theoretical point of view since they are described by (infinite-dimensional) functional equations and because of its interest in applications like, for instance, population growth models, transportation, communications as well as war–peace and agricultural models [1,9]). It exists a wide variety of both dependent on the delays and independent of the delays results (see, for instance, [3–5,7,11,12]) obtained via Lyapunov stability theory or frequency-domain analysis tools, the second one being only useful in the time-invariant case. Most of the available results are restricted to time-invariant systems with constant delays. However, nowadays, the extensions to the nonlinear and time-varying systems as well as to systems described by partial derivative equations are receiving an increasing interest in the literature (see, for instance, [4–6,8,10]). In this paper, a very general class of time-varying systems is considered whose delay is time-varying but not necessarily everywhere bounded time-differentiable while the delay-free and delayed dynamics are, in general, time-varying and impulsive, and whose external input is impulsive as well what is of interest in some applications. Three different unforced auxiliary systems related to the current unforced delay dynamic system are first introduced to facilitate the analysis. The first of such unforced auxiliary systems is a delay-free time-invariant one whose evolution operator is a C_0-semigroup with an infinitesimal generator. The second one is a delay-free (in general, time-varying) homogeneous dynamic system which contains the delay-free dynamics of the whole current system which is, in general, impulsive and its parameterization is subject to bounded discontinuities of first and second class. Its evolution operator is bounded and almost everywhere time-differentiable. Finally, the third auxiliary system contains all the dynamics of the unforced time-delay system under study. Its bounded evolution operator involves explicitly the delay function and it is almost everywhere time-differentiable. The relevant associate state–state/output and input–state/output operators defined to build the trajectory solutions are characterized from the above evolution operators and their associate evolution equations. The auxiliary systems are then used to build explicitly in three different ways the unique state and output solution trajectories for any given admissible initial conditions which is then used to obtain a set of global stability results for the dynamic system via Gronwall's lemma or Lyapunov's stability theory [1,13], by taking into account that the system parameterization possesses bounded discontinuities and impulsive terms [11,12], while the input might impulsive as well, [11]. The last part of the manuscript is devoted to investigate the compactness of the various operators (see, for instance, [14,15] for definitions and relevant properties) mapping the input into state and output Banach's spaces if the input is square-integrable on $[0, \infty)$. Some results are presented for the so-called gate-operators being relevant in some applications of electronics and signal theory [15]. The main motivation is the relevance of compact operators in the approximation theory in Banach's (and then in Hilbert) spaces since they map bounded sets into totally bounded sets (since any operator is compact in a reflexive space if and only if it is completely continuous). Also, compact operators in topological or metric spaces map weakly convergent sequences into strongly convergent ones. Thus, the solution trajectories are either finite-dimensional or arbitrarily close to finite dimension functions if the input is square-integrable on $[0, \infty)$ and the input–state, respectively, input–output operator is compact.

2. Notation

- $\mathbf{R}_0^+(\mathbf{Z}_0^+) = \mathbf{R}^+ \cup \{0\}(\mathbf{Z}^+ \cup \{0\})$ and $\mathbf{R}_0^-(\mathbf{Z}_0^-) = \mathbf{R}^- \cup \{0\}(\mathbf{Z}^- \cup \{0\})$ are the sets of nonnegative and negative real (integer) numbers in the real field $\mathbf{R}$ (integer ring $\mathbf{Z}$). The complement of a subset $S \subseteq \mathbf{R}$ in $\mathbf{R}$ is denoted as $\bar{S}$. The exponential function of "f" is denoted indistinctly as "$\exp(f)$" or "e^f" with the main criterion of notation choice being reading quality.

$\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote respectively, the maximum and minimum eigenvalue of the square real (symmetric) matrix $P = P^T$. The notation $P > Q$ ($P \geqslant Q$) means that $P - Q$ is positive definite denoted by $P - Q > 0$ ($P - Q$ being positive semidefinite is denoted by $P - Q \geqslant 0$) provided that Q is symmetric of the same order as P. I is the identity matrix of any order (depending on context) or specified as a subscript if necessary.

- $X \subseteq \mathbf{R}^n$, $U \subseteq \mathbf{R}^m$ and $Y \subseteq \mathbf{R}^p$ are, respectively, the state, input and output real spaces of the time-delay dynamic system of respective dimensions n, m and p so that the state, input and output real vectors are, respectively, X, U and Y for all $t \geqslant 0$. The same delta-symbol with integer subscripts, i.e. δ_{ij} is unity if and only if $i = j$ and zero, otherwise, instead of a time argument will be used for the Kronecker delta through Section 5.
- The real n-vector function $\varphi = \varphi_0 + \tilde{\varphi} + \tilde{\varphi}_{\text{imp}}$ of the time-delay system is the set of admissible initial conditions $IC([-\bar{r}, 0], \mathbf{R}^n)$ where $\infty > \bar{r} \geqslant r(0) \geqslant 0$ and the time-varying delay function $r : \mathbf{R}_0^+ \to \mathbf{R}_0^+$ so that $\varphi_0 : [-\bar{r}, 0) \to \mathbf{R}^n$ is absolutely continuous, $\tilde{\varphi}(t) = \tilde{\varphi}(t_i^+)$, $\forall t \in (t_i, t_{i+1})$ with $\tilde{\varphi}(t_i^-) \neq \tilde{\varphi}(t_i^+)$ being isolated bounded jump discontinuities at $t_i \in [-\bar{r}, 0] \cap \mathbf{R}_0^+$ with $\max_{t_i \in \operatorname{supp}(\tilde{\varphi})}(\|\tilde{\varphi}(t_i^+)\|) \leqslant M < \infty$. The subsequent study remains essentially valid if $\tilde{\varphi}(t) = 0$, $\forall t \in (t_i, t_{i+1})$. A notation convention adopted is that at any discontinuity points t_i, $\tilde{\varphi}(t_i^+)$ ($\tilde{\varphi}(t_i^-)$) denotes the value of $\varphi(t_i)$ to the right (to the left) of t_i. $\tilde{\varphi}_{\text{imp}}(t) = \sum_{t_i \in TN_{\text{imp}}} \tilde{\varphi}_{\text{imp}}(t_i)\delta(t - t_i)$ is a real n-vector function from $[-\bar{r}, 0) \cap \mathbf{R}^+$ to $\mathbf{R}^n$ taking nonzero values at set $TN_{\text{imp}} \subset [-\bar{r}, 0) \cap \mathbf{R}^+$ of finite cardinal and zero measure with $\delta(t)$ being the Dirac distribution function; i.e.

$$\lim_{t \to \alpha}\left(\int_{\sigma - t}^{\sigma + t} g(\sigma - \tau)\delta(\tau)\, d\tau\right) = g(\sigma)$$

  for any real $\alpha \in (0, \infty)$.
- L_∞^m is the set of essentially bounded m-vector functions from $\mathbf{R}_0^+$ to $\mathbf{R}_0^+$. $L_2^m(a, b) \equiv L_2((a, b), \mathbf{R}^m)$ is the Hilbert space of the real m-vector functions $f : (a, b) \to \mathbf{R}^m$ which are square-integrable on (a, b) with the inner product denoted by $\langle \cdot, \cdot \rangle$ and the (semi)norm

$$\|f\|_{L_2^m(a,b)} := \langle f, f \rangle_{L_2^m(a,b)}^{1/2}, \quad \forall f \in L_2^m(a, b).$$

  L_2^m denotes $L_2^m(0, \infty) \equiv L_2^m((0, \infty), \mathbf{R}^m)$. The notation $M \in L_2^{n \times m} \equiv L_2((0, \infty), \mathbf{R}^{n \times m})$ is used for any matrix function $M = (m_{ij})$ of real entries m_{ij} in L_2. Since impulsive functions are widely used thoroughly, closed and one-side closed real intervals $[a, b]$ (respectively $(a, b]$, $[a, b)$) are used when necessary, as well as related simplified notations for Lebesgue integrals involving the appropriate limits to the left/right as follows: $\int_{a^+}^{b^+}(\cdot)$, $\int_{a^-}^{b^-}(\cdot)$, $\int_{a^+}^{b^-}(\cdot)$, $\int_{a^-}^{b^+}(\cdot)$. For instance, if $f(t) = g(t) + k_a\delta(t - a) + k_b\delta(t - b)$ is a real function of domain $(a, b) \cap \mathbf{R}$ with $\delta(t)$ being the Dirac distribution and g being a real Lebesgue integrable function on (a, b), then

$$\int_{a^-}^{b^+} f(\tau)\, d\tau = \int_a^b g(\tau)\, d\tau + k_a + k_b, \qquad \int_{a^+}^{b^+} f(\tau)\, d\tau = \int_a^b g(\tau)\, d\tau + k_b,$$

$$\int_{a^-}^{b^-} f(\tau)\, d\tau = \int_a^b g(\tau)\, d\tau + k_a \quad \text{and} \quad \int_{a^+}^{b-} f(\tau)\, d\tau = \int_a^b g(\tau)\, d\tau.$$

- The set of real absolutely integrable m-vector functions of domain $(a,b)\cap\mathbf{R}$ is denoted by $L_1^m(a,b)$ with $L_1^m\equiv L_1^m(0,\infty)=L_1((0,\infty),\mathbf{R}^m)$. A (truncated) function $f_t(\tau)$ of $f(\tau)$ on $[0,t]\subset\mathbf{R}$ is defined as $f_t(\tau):=f(\tau)[\mathbf{1}(\tau)-\mathbf{1}(\tau-t)]$, which equates $f(\tau)$ on $[0,t]$ and is zero for $\forall\tau\notin[0,t]$, where $\mathbf{1}(t)=1$ for all $t\geqslant 0$ and $\mathbf{1}(t)=0$ for $t<0$ is the unity step (Heaviside) function. The space of truncated square-integrable real m-vector functions on $(0,\infty)$ is denoted by $L_{2\mathrm{e}}^m\equiv L_{2\mathrm{e}}^m((0,\infty),\mathbf{R}^m)$, defined as

$$L_{2\mathrm{e}}^m:=\left\{f_t\in L_2^m,\ \text{for all finite } t\geqslant 0\right\}=\bigcup_{0<t<\infty}L_2^m(0,t)$$

endowed with an inner product $\langle\cdot,\cdot\rangle_{L_{2\mathrm{e}}^m}^{1/2}:=\sup_{0\leqslant t<\infty}(\int_{-\infty}^{\infty}f_t^T(\tau)f_t(\tau)\,d\tau)$ and associate (semi)norm

$$\|f\|_{L_{2\mathrm{e}}^m}:=\langle f,f\rangle_{L_{2\mathrm{e}}^m}^{1/2}\quad\forall f\in L_{2\mathrm{e}}^m$$

(or, equivalently, $\forall f_t\in L_2^m$ for all finite $t\geqslant 0$). The formal usefulness of the space $L_{2\mathrm{e}}^m$ is that truncated functions of non-square integrable functions are square-integrable, in general. Thus, if $f\notin L_2^m$, but $f_t\in L_2^m$ for all $0\leqslant t<\infty$, then $f\in L_{2\mathrm{e}}^m$ and most of the properties of the Hilbert space L_2^m may be invoked for f (via f_t) for all finite real intervals $[0,t]$. The notation $f\in L_2^m$ may be further specified as $f\in L_2((0,\infty)\cap\mathbf{R},F\subseteq\mathbf{R}^m)$ to indicate that $f\in F$ for a real f with definition domain $[0,\infty)$. Similarly, $f\in L_{2\mathrm{e}}^m$ if $f_t\in F$ for all finite $t\geqslant 0$.
- The space $L_{1\mathrm{e}}^m$ of (truncated) absolutely integrable real m-vector functions is defined in an analogous away related to the space

$$L_1^m\equiv L_1\big((0,\infty),\mathbf{R}^m\big)=\left\{f:[0,\infty)\cap\mathbf{R}\to\mathbf{R}^m\colon\int_0^\infty\big(f^T(\tau)f(\tau)\,d\tau\big)^{1/2}<\infty\right\}.$$

- $x_{[t]}$ is a strip of the solution trajectory; i.e. $x_{[t]}\equiv x:[t-r(t),t]\to\infty$ for $t\geqslant 0$ of the dynamic time-delay system of point time-varying delay $r(t)$.
- The set of linear operators Γ from the linear space X to the linear space Y is denoted by $\mathbf{L}(X,Y)$.
- The same norm symbol $\|\cdot\|$ is used for vector and (induced) matrix norms in Euclidean spaces as that used for the spaces L_∞^m, L_2^m and L_1^m; i.e. if $z\in\mathbf{R}^m$ and $Z\in\mathbf{R}^{n\times m}$ then

$$\|Z\|:=\sup_{z\neq 0}\left(\frac{\|Zz\|}{\|z\|}\right)=\sup_{\|z\|\leqslant 1}\big(\|Zz\|\big)$$

for any (vector) norm. Each particular norm symbol is interpreted without difficulty depending on context. When necessary for clarity, the norm symbol is appropriately subscripted or described. Linear operators $Z\in\mathbf{L}(L_2^p,L_2^q)$ from the Banach's space L_2^p to the Banach's space L_2^q are usually defined point-wise as $(Zf)(t):[0,t]\times\mathbf{R}^p\to\mathbf{R}^q$ for each $f\in L_2^p$ in the definition domain of the M-operator. The norm of the M-operator is

$$\begin{aligned}\|Z\|&:=\sup_{\|f\|=1}\left(\frac{\|Zf\|}{\|f\|}\colon\ f\in L_2^p\right)\\&=\left\{\inf k\in\mathbf{R}_0^+\colon\ \|Zf\|\leqslant k,\ \forall f\in L_2^p\text{ such that }\|f\|\leqslant 1\right\}\end{aligned}$$

while its adjoint is $Z^*\in\mathbf{L}(L_2^q,L_2^p)$ with norm defined accordingly.

- Indicator binary functions with domain $\mathbf{R}_0^+$ are used to evaluate time-integral functions containing impulses. For instance, for $t_i \in TN(0,t)$ with $h(t) := \int_0^t f(\tau)\,d\tau$ where $f:[0,t] \to \mathbf{R}$ is Lebesgue integrable,

$$\int_{0^-}^{t^-} \Bigl(f(\tau) + \sum_{t_i \in TN} g(\tau)\delta(\tau - t_i) \Bigr) d\tau = h(t) + \sum_{t_i \in TN(0,t)} g(t_i)$$

$$= h(t) + \int_{0^-}^{t^-} \mu(\tau) g(\tau) \delta(\tau - t_i)\, d\tau$$

$$= \int_{0^-}^{t^-} \bigl(f(\tau) + \mu(\tau) g(\tau) \delta(\tau - t_i) \bigr)\, d\tau,$$

where $TN(0,t) := \{\tau \in TN\colon\ 0 \leqslant \tau < t\}$ is the support (of zero Lebesgue measure) of the real function $g:[0,t) \to \mathbf{R}$, provided that $\sum_{t_i \in TN(0,t)} g(t_i) < \infty$, while $\mu:(0,t) \to \{0,1\}$ is a (binary) indicator function of $TN(0,t)$ defined as $\mu(t) = 0$ if $t \notin TN(0,t)$ and $\mu(t) = 1$ if $t \in TN(0,t)$.

3. Time-variant time-delay differential system

Consider the dynamic system:

$$\text{S:}\ \dot{x}(t) = A(t)x(t) + A_{\mathrm{d}}x(t - r(t)) + B(t)u(t), \tag{1a}$$

$$y(t) = C(t)x(t) + D(t)u(t), \tag{1b}$$

where u, x and y are the input, state and output real vector functions satisfying $u \in L_2((0,\infty), U) \subset L_2^m$, $x \in L_{2\mathrm{e}}((0,\infty), X) \subset L_{2\mathrm{e}}^n$ and $y \in L_{2\mathrm{e}}((0,\infty), Y) \subset L_{2\mathrm{e}}^p$, respectively; and U, X and Y are the m-dimensional real input, n-dimensional real state and p-dimensional real output linear spaces, respectively: i.e. $u(t) \in U$, $x(t) \in X$ and $y(t) \in Y$ for all $t \geqslant 0$. The system (1) is subject to any function of initial conditions $\varphi \in IC([-\bar{r}, 0], \mathbf{R}^n)$, of the form defined in Section 2, where $r(t)$ is a (non-necessarily bounded) time-delay function $r:[0,\infty) \to \mathbf{R}_0^+$ satisfying $0 \leqslant r(t) \leqslant t + \bar{r}$ $(0 \leqslant r(0) \leqslant \bar{r} < \infty)$ for all $t \in \mathbf{R}_0^+$, and

$$A(t) := A'(t) + \sum_{t_i \in TN} A''(t)\delta(t - t_i) = A_0 + \tilde{A}'(t) + \sum_{t_i \in TN} A''(t)\delta(t - t_i), \tag{2a}$$

$$A_{\mathrm{d}}(t) := A'_{\mathrm{d}}(t) + \sum_{t_i \in TN_{\mathrm{d}}} A''_{\mathrm{d}}(t)\delta(t - t_i) \tag{2b}$$

are, in general, impulsive delay-free and delayed real matrix functions of dynamics from $[0,\infty)$ to $\mathbf{R}^{n \times n}$, respectively, where DD and $D\overline{D}$ are, respectively, discrete real subsets of time instants where the time-delay function is discontinuous and continuous non-differentiable, respectively, while CD and $C\overline{D}$ denote, respectively, the real and discrete real subsets of time instants where it is continuously differentiable and continuous non-differentiable. $A_0 \in \mathbf{R}^{n \times n}$ is a constant real n-matrix. $A'(t)$, $\tilde{A}'(t) := A'(t) - A_0$ and $A'_{\mathrm{d}}(t)$ have piecewise bounded continuous entries with isolated jump bounded discontinuities at time instants TD and TD_{d}, respectively. $A''(t)$ and $A''_{\mathrm{d}}(t)$ are bounded matrix functions from $[0,\infty)$ to $\mathbf{R}^{n \times n}$ of support of zero Lebesgue measure consisting of the set of impulses located at the time instants TN and TN_{d}, respectively. By convenience for evaluation of time integrals, continuous binary indicator real functions $\mu : [0,\infty) \to \{0,1\}$ and $\mu_{\mathrm{d}} : [0,\infty) \to \{0,1\}$ will be used when necessary defined as $\mu(t) = 1$ ($\mu_{\mathrm{d}}(t) = 1$) if $t \in TN$ ($t \in TN_{\mathrm{d}}$); and $\mu(t) = 0$ ($\mu_{\mathrm{d}}(t) = 0$) if $t \notin TN$ ($t \notin TN_{\mathrm{d}}$). $B : [0,\infty) \to \mathbf{R}^{m\times n}$; $C : [0,\infty) \to \mathbf{R}^{n\times p}$ and $D : [0,\infty) \to \mathbf{R}^{p\times p}$ are, respectively, the control, output and interconnection real matrix functions of continuous bounded entries.

The input u in $L_2([0,\infty), U \subseteq \mathbf{R}^m)$ may be also impulsive and to possess discontinuities of second class (so-called jump discontinuities) at the set TU; i.e.

$$u(t) = u'(t) + \sum_{t_i \in TU} u''(t)\delta(t - t_i) \quad \text{with}$$

$$u \in L_2^m,\ u' \in L_2^m \quad \text{and} \quad t_{i+1} - t_i \geqslant T_{0u} > 0, \quad \mu_u : [0,\infty) \to \{0,1\}$$

and defined as $\mu_u(t) = 1$ if $t \in TU$ and $\mu_u(t) = 0$ ($t \notin TU$) is a binary indicator of TU. Note that DD, $D\overline{D}$, $C\overline{D}$, TN, TD, TD_{d}, TN_{d}, TU are strictly ordered sets in $\mathbf{R}_0^+$ and, respectively, in $\mathbf{Z}_0^+$ with respect to the "less than" binary relation "$<$" (i.e. satisfying the anti-reflexive, anti-symmetric and transitive properties). The notation $D(a,b) := \{t_i \in D\colon\ a \leqslant t_i < b\}$ for given $a, b \in \mathbf{R}_0^+$ applies to fix elements of any set D (being, in particular, DD, $D\overline{D}$, TN, TD, TD_{d}, TN_{d} or TU) in $[a,b) \cap \mathbf{R}_0^+$.

3.1. The use and usefulness of the indicator sets and functions

When considering the state-trajectory solution of (1a) for $t \geqslant 0$ subject to initial conditions $\varphi \in IC([-\bar{r},0],\mathbf{R}^n)$, note that

$$\begin{aligned}
\dot{x}(t^-) &= A'(t^-)x(t^-) + A'_{\mathrm{d}}(t^-)x\big(t^- - r(t)\big) + B(t)u'(t^-),\\
x(t^+) &= \big(I + \mu(t)A''(t)\big)x(t^-) + \mu_{\mathrm{d}}(t)A''_{\mathrm{d}}(t)x\big(t^- - r(t)\big) + \mu_u(t)B(t)u''(t),\\
\dot{x}(t^+) &= A'(t^+)\Big(\big(I + \mu(t)A''(t)\big)x(t^-) + \mu_{\mathrm{d}}(t)A''_{\mathrm{d}}(t)x\big(t^- - r(t)\big) + \mu_u(t)B(t)u''(t)\Big)\\
&\quad + A'_{\mathrm{d}}(t^+)x\big(t^+ - r(t)\big) + B(t)u'(t^+),
\end{aligned}$$

so that $x(t^+) = x(t^-)$ if $\mu(t) = \mu_{\mathrm{d}}(t) = \mu_u(t) = 0$; i.e. if $t \notin (TN \cup TN_{\mathrm{d}} \cup TU)$. Another useful notation with indicators is

$$x\big(t_i^+\big) = \begin{cases} x(t_i^-) \quad \text{if } t_i \notin (TN \cup TN_{\mathrm{d}} \cup TU), \quad \text{and}\\ (I + A''(t_i))x(t_i^-) + A''_{\mathrm{d}}(t_i)x(t_i^- - r(t)) + B(t_i^+)u''(t_i)\\ \quad \text{if } t_i \in (TN \cap TN_{\mathrm{d}} \cap TU). \end{cases}$$

The sets TN, TN_{d}, TU, etc., are useful to include the contribution of the impulses to the dynamics through time. For instance, assume zero initial conditions; i.e. $\varphi \equiv 0$, then the (forced) solution trajectory of (1a) at t^- may be expressed as

$$x(t^-) = \int_{0^-}^{t^-} e^{A_0(t-\tau)}\big[A_{\mathrm{d}}(\tau)x\big(\tau - r(\tau)\big) + B(\tau)u'(\tau)\big]\,d\tau + \sum_{t_i \in TU(0,t)} e^{A_0(t-t_i)}B(t_i)u''(t_i).$$

The last right-hand side may also be denoted by using the sum over the indicator $\mathrm{I}(TU(0,t))$ as

$$\sum_{t_i \in TU(0,t)} e^{A_0(t-t_i)}B(t_i)u''(t_i)$$

or, alternatively, it may be included in the integrand as

$$e^{A_0(t-\tau)}\mu_u(\tau)B(\tau)\delta(\tau)u''(\tau)\,d\tau$$

by using the binary indicator function $\mu_u : \mathbf{R}_0^+ \to \{0, 1\}$. The choice of each notation is made according to convenience criteria.

3.2. Auxiliary homogeneous dynamic systems

It is now discussed how the unique state and output trajectories $x \in L_{2e}^n$ and $y \in L_{2e}^p$ may be equivalently built from three different homogeneous (i.e. unforced) dynamic systems for each $\varphi \in IC([-\bar{r}, 0], \mathbf{R}^n)$, two being delay-free, with one of them being in addition time-invariant, while the third one is the unforced (1a). This allows to highlight the decomposition of the trajectory solutions into parts and then to discuss stability results based on different conditions and assumptions from (1) as well as the compactness of the relevant operators associated with the trajectories. The three auxiliary homogeneous systems are:

S1: $\dot{z}_{A0}(t) = A_0 z_{A0}(t)$, $z_{A0}(0) = z_0 \in X \subset \mathbf{R}^n$, for any arbitrary constant matrix $A_0 \in \mathbf{R}^{n\times n}$ such that $A'(t) = A_0 + \tilde{A}'(t)$; $A(t) = A_0 + \tilde{A}(t)$, with $\tilde{A}'(t)$ prescribed after fixing A_0 and $\tilde{A}(t) = \tilde{A}'(t) + A''(t)$.
S2: $\dot{z}_A(t) = A(t)z_A(t)$, $z_A(0) = z_0 \in X \subset \mathbf{R}^n$.
S3: $\dot{z}(t) = A(t)z(t) + A_d(t)z(t - r(t))$, $z(0) = z_0 \in X \subset \mathbf{R}^n$.

Note that S3 is the unforced system (1a) and also a forced system with the forcing term $A_d(t)z_A(t - r(t))$. Also, $(A(t) - A_0)z_{A0}(t) + A_d(t)z_{A0}(t - r(t))$ is a forcing term generating the solution of S3 from that of the homogeneous S1. S1 is an unforced delay-free time invariant system which may be taken as a reference value for the delay-free dynamics. The stability of (1a) may be formulated with respect to a stability matrix A_0 or A_0 may be the delay-free average delay-free dynamics in the case when $A'(t)$ is slowly time-variant. S2 is an unforced delay-free, in general, time variant system which becomes identical to the unforced (1a) when the delayed dynamics is identically zero for all time.

3.3. Main result of Section 3

The following result holds concerning the unique state/output trajectory of (1) from the auxiliary systems S1, S2 on $\mathbf{R}_0^+$ with identical initial conditions $z(0)$, $z_{A0}(0)$, $z_A(0)$ equating $\varphi_0(0) + \tilde{\varphi}(0^-)$ with S3, subject to $z(t) \equiv \varphi(t)$ for all $t \in [-\bar{r}, 0] \cap \mathbf{R}_0^+$ with any $\varphi \in IC([-\bar{r}, 0], \mathbf{R}^n)$ satisfying the above constraints at $t = 0$. The following result holds.

Theorem 1. *The unique state/output trajectories of* S, *equations* (1), *on* $\mathbf{R}_0^+$ *such that* $z(t) \equiv \varphi(t)$ *for* $t \in [-\bar{r}, 0] \cap \mathbf{R}_0^+$ *and* $r(t) \in [0, t + \bar{r}]$ *are uniquely defined for all* $t \geqslant 0$, *by any of the three sets of evolution equations below*:

(i) *Evolution Equations 1* (*EE1*) *from* S1.

$$x(t^-) = (S_{A0}\varphi)(t) + (S'_{A0}x_{[t]})(t^-) + \big(S_{A0}^u u\big)(t^-) \tag{3a}$$

$$= (\bar{S}_{A0}\bar{x}_0)(t) + (\bar{S}'_{A0}x_{[t]})(t^-) + \big(S_{A0}^u u\big)(t^-), \tag{3b}$$

$$x(t^+) = (S_{A0}\varphi)(t) + (S'_{A0}x_{[t]})(t^+) + \big(S^u_{A0}u\big)(t^+) \tag{3c}$$

$$= (\bar S_{A0}\bar x_0)(t) + (\bar S'_{A0}x_{[t]})(t^+) + \big(\bar S^u_{A0}u\big)(t^+) \tag{3d}$$

$$= \big(I + \mu(t)A''(t)\big)x(t^-) + \mu_{\mathrm d}(t)A''_{\mathrm d}(t)x\big(t^- - r(t)\big) + \mu_u(t)B(t)u''(t),$$

$$y(t^-) = (M_{A0}\varphi)(t) + (M'_{A0}x_{[t]})(t^-) + \big(M^u_{A0}u\big)(t^-) \tag{3e}$$

$$= (\bar M_{A0}\bar x_0)(t) + (\bar M'_{A0}x_{[t]})(t^-) + \big(M^u_{A0}u\big)(t^-), \tag{3f}$$

$$y(t^+) = (M_{A0}\varphi)(t) + (M'_{A0}x_{[t]})(t^+) + \big(M^u_{A0}u\big)(t^+) \tag{3g}$$

$$= (\bar M_{A0}\bar x_0)(t) + (\bar M'_{A0}x_{[t]})(t^+) + \big(\bar M^u_{A0}u\big)(t^+) \tag{3h}$$

$$= C(t)\Big(\big(I + \mu(t)A''(t)\big)x(t^-) + \mu_{\mathrm d}(t)A''_{\mathrm d}(t)x\big(t^- - r(t)\big) + \mu_u(t)B(t)u''(t)\Big) \tag{3i}$$

$$+ \mu_u(t)D(t)u''(t)\delta(0), \tag{3j}$$

where $S_{A0} \in \mathbf{L}(IC, L^n_{2\mathrm e})$, $S'_{A0} \in \mathbf{L}(L^n_{2\mathrm e}, L^n_{2\mathrm e})$, $\bar S_{A0} \in \mathbf{L}(L^m_2, L^n_{2\mathrm e})$, $\bar S'_{A0} \in \mathbf{L}(L^n_{2\mathrm e}, L^n_{2\mathrm e})$, $S^u_{A0} \in \mathbf{L}(L^m_2, L^n_{2\mathrm e})$, $M_{A0} \in \mathbf{L}(IC, L^p_{2\mathrm e})$, $M'_{A0} \in \mathbf{L}(L^n_{2\mathrm e}, L^p_{2\mathrm e})$ *and* $M^u_{A0} \in \mathbf{L}(L^m_2, L^p_{2\mathrm e})$ *are defined point-wise at* t^- *via*:

$$(S_{A0}\varphi)(t) := e^{A_0 t}\left(x_0^+ + \int_{I_{1t}} e^{-A_0\tau}A_{\mathrm d}(\tau)\varphi\big(\tau - r(\tau)\big)\,d\tau\right),$$

$$x(0^+) = x_0^+ = \varphi_0(0) + \tilde\varphi(0^+), \tag{4a}$$

$$(S'_{A0}x_{[t]})(t^-) := e^{A_0 t}\left[\int_{I_{2t}} e^{-A_0\tau}A_{\mathrm d}(\tau)x\big(\tau - r(\tau)\big)\,d\tau + \int_{0^-}^{t^-} e^{-A_0\tau}\tilde A(\tau)x(\tau)\,d\tau + \int_{\bar r}^{t^-} e^{-A_0\tau}A_{\mathrm d}(\tau)x\big(\tau - r(\tau)\big)\big[\mathbf 1\big(\tau - r(\tau)\big)\big(\mathbf 1(\tau) - \mathbf 1(\tau - t)\big)\big]\,d\tau\right], \tag{4b}$$

$$(\bar S_{A0}\bar x_0)(t) := e^{A_0 t}\bar x_0,$$

$$\bar x_0 := x_0^+ + \int_{0^-}^{\bar r} e^{-A_0\tau}A_{\mathrm d}(\tau)\varphi\big(\tau - r(\tau)\big)\big[\mathbf 1(\tau) - \mathbf 1\big(\tau - r(\tau)\big)\big]\,d\tau, \tag{4c}$$

$$(\bar S'_{A0}\bar x_0)(t^-) := e^{A_0 t}\left[\int_{0^-}^{t^-} e^{-A_0\tau}\tilde A(\tau)x(\tau)\,d\tau + \int_{I_{2t}} e^{-A_0\tau}A_{\mathrm d}(\tau)x\big(\tau - r(\tau)\big)\,d\tau - \int_{I'_{1t}} e^{-A_0\tau}A_{\mathrm d}(\tau)x\big(\tau - r(\tau)\big)\,d\tau + \int_{\bar r}^{t^-} e^{-A_0\tau}A_{\mathrm d}(\tau)x\big(\tau - r(\tau)\big)\big[\mathbf 1\big(\tau - r(\tau)\big)\big(\mathbf 1(\tau) - \mathbf 1(\tau - t)\big)\big]\,d\tau\right], \tag{4d}$$

where $I_t := (I_{1t} \cup I_{2t}) \cap \mathbf R = (0, \min(\bar r, t) \cap \mathbf R)$ *is a real interval of measure* $\min(\bar r, t)$ *for each* $t \geqslant 0$,

$$I'_{1t} := \big([0,\bar r]\cap\mathbf{R}\big)/I_{1t} = \big\{\tau\in\mathbf{R}_0^+:\ \bar r \geqslant \tau > r(\tau)\big\} \quad (\equiv I_{2t} \textit{ if } 0\leqslant t\leqslant \bar r),$$
$$I_{1t} := \big\{\tau\in\big[0,\min(\bar r,t)\big]\cap\mathbf{R}:\ \tau\leqslant r(\tau)\big\},$$
$$I_{2t} := \big\{\tau\in\big[0,\min(\bar r,t)\big]\cap\mathbf{R}:\ \tau > r(\tau)\big\} = \big[0,\min(\bar r,t)\big]\cap\mathbf{R}/I_{1t}, \tag{4e}$$

$$\big(\bar S^u_{A0}u\big)(t^-) := \int_{0^-}^{t^-} e^{A_0(t-\tau)}B(\tau)u(\tau)\,d\tau$$
$$= \int_{0^-}^{t^-} e^{A_0(t-\tau)}B(\tau)u(\tau)\,d\tau + \sum_{t_i\in TU(0,t)} e^{-A_0(t-t_i)}B(t_i)u''(t_i), \tag{4f}$$

$$(M_{A0}\varphi)(t) := C(t)(S_{A0}\varphi)(t),$$
$$(M'_{A0}x_{[t]})(t) := C(t)(S'_{A0}x_{[t]})(t^-), \tag{4g}$$
$$(\bar M_{A0}\bar x_0)(t) := C(t)(\bar S_{A0}\bar x_0)(t),$$
$$(\bar M'_{A0}x_{[t]})(t^-) := C(t)(\bar S'_{A0}x_{[t]})(t^-), \tag{4h}$$
$$\big(M^u_{A0}u\big)(t^-) := C(t)\big(S^u_{A0}u\big)(t^-) + D(t)u'(t^-), \tag{4i}$$

and $(S'_{A0}x_{[t]})(t^+)$, $(\bar S'_{A0}x_{[t]})(t^+)$, $(M'_{A0}x_{[t]})(t^+)$, $(\bar M'_{A0}x_{[t]})(t^+)$, $(S^u_{A0}u)(t^+)$ *and* $(M^u_{A0}x_{[t]})(t^+)$ *are defined similarly as their counterparts at* t^- *by replacing* t^- *with* t^+ *in the corresponding definitions.*

(ii) *Evolution Equations 2* (*EE2*) *from* S2.

$$x(t^-) = (S_A\varphi)(t) + (S'_A x_{[t]})(t^-) + \big(S^u_A u\big)(t^-), \tag{5a}$$
$$x(t^+) = (S_A\varphi)(t^+) + (S'_A x_{[t]})(t^+) + \big(S^u_A u\big)(t^+)$$
$$= \big(I+\mu(t)A''(t)\big)x(t^-) + \mu_{\mathrm d}(t)A''_{\mathrm d}(t)x\big(t^- - r(t)\big) + \mu_u(t)B(t)u''(t), \tag{5b}$$
$$y(t^-) = (M_A\varphi)(t^-) + (M'_A x_{[t]})(t^-) + \big(M^u_A u\big)(t^-)$$
$$= C(t)\big((S_A\varphi)(t^-) + (S'_A x_{[t]})(t^-) + \big(S^u_A u\big)(t^-)\big) + D(t)u'(t^-), \tag{5c}$$
$$y(t^+) = (M_A\varphi)(t^+) + (M'_A x_{[t]})(t^+) + \big(M^u_A u\big)(t^+)$$
$$= C(t)\big((S_A\varphi)(t^+) + (S'_A x_{[t]})(t^+) + \big(S^u_A u\big)(t^+)\big) + D(t)u'(t^+)$$
$$+ \mu_u(t)D(t)u''(t)\delta(0)$$
$$= C(t)\big(\big(I+\mu(t)A''(t)\big)x(t^+) + \mu_{\mathrm d}(t)A''_{\mathrm d}(t)x\big(t^+ - r(t)\big) + \mu_u(t)B(t)u''(t)\big)$$
$$+ D(t)u'(t^+) + \mu_u(t)D(t)u''(t)\delta(0), \tag{5d}$$

where the linear operators $S_A\in\mathbf{L}(L^n_{2\mathrm e},L^n_{2\mathrm e})$, $S'_A\in\mathbf{L}(L^n_{2\mathrm e},L^n_{2\mathrm e})$, $S^u_A\in\mathbf{L}(L^m_2,L^n_{2\mathrm e})$, $M_A\in\mathbf{L}(IC,L^p_{2\mathrm e})$, $M'_A\in\mathbf{L}(L^n_{2\mathrm e},L^p_{2\mathrm e})$ *and* $M^u_A\in\mathbf{L}(L^m_2,L^p_{2\mathrm e})$ *are defined point-wise at* t^- *via*:

$$(S_A\varphi)(t^-) := \Psi_A(t^-,0)\left(x_0^+ + \int_{0^-}^{\bar r}\Psi_A(t^-,\tau)A_{\mathrm d}(\tau)\varphi\big(\tau-r(\tau)\big)\big(\mathbf{1}(\tau)-\mathbf{1}(\tau-t)\big)\,d\tau\right),$$
$$x(0^+) = \varphi_0(0) + \tilde\varphi(0^+), \tag{6a}$$

$$(S'_A x_{[t]})(t^-) := \left[\int_{\bar{r}}^{t^-} \Psi_A(t^-,\tau) A_{\mathrm{d}}(\tau) x\big(\tau - r(\tau)\big) \mathbf{1}(t-\tau)\, d\tau \right.$$
$$+ \int_{I_{2t}} \Psi_A(t^-,\tau) A_{\mathrm{d}}(\tau) x\big(\tau - r(\tau)\big)\, d\tau$$
$$\left. - \int_{I_{1t}}^{t^-} \Psi_{A'}(t^-,\tau) A_{\mathrm{d}}(\tau) x\big(\tau - r(\tau)\big)\, d\tau \right], \tag{6b}$$

$$\big(\bar{S}_A^u u\big)(t^-) := \int_{0^-}^{t^-} \Psi_A(t^-,\tau) B(\tau) u(\tau)\, d\tau$$
$$= \int_{0^-}^{t^-} \Psi_A(t^-,\tau) B(\tau) u'(\tau)\, d\tau + \sum_{t_i \in TU(0,t)} \Psi_A(t^-,t_i) B(t_i) u''(t_i) \tag{6c}$$

$$\big(\bar{S}_A^u u\big)(t^+) := \int_{0^-}^{t^+} \Psi_A(t^+,\tau) B(\tau) u(\tau)\, d\tau$$
$$= \big(I + \mu(t) A''(t)\big)(S_u u)(t^-) + \mu_u(t) B(t) u''(t), \tag{6d}$$

$$(M_A \varphi)(t^-) := C(t)(S_A \varphi)(t^-),$$
$$(M'_A x_{[t]})(t^-) := C(t)(S'_A x_{[t]})(t^-), \tag{6e}$$
$$\big(M_A^u u\big)(t^-) := C(t)\big(S_A^u u\big)(t^-) + D(t) u'(t^-), \tag{6f}$$
$$\big(M_A^u u\big)(t^+) := C(t)\big(S_A^u u\big)(t^+) + D(t) u'(t^+) + \mu_u(t) D(t) u''(t), \tag{6g}$$

and $(S_A\varphi)(t^+)$, $(S'_A x_{[t]})(t^+)$, $(M'_A x_{[t]})(t^+)$ *and* $(S_A^u u)(t^+)$ *are defined similarly as their counterparts at* t^- *by replacing* t^- *with* t^+ *in the corresponding definitions and the evolution operator* $\Psi_A(t^-,0)$ *satisfies the first-order differential system*:

$$\dot{\Psi}_A(t^-,0) = A(t^-)\Psi_A(t^-,0) = A'(t^-)\Psi_A(t^-,0) \quad \textit{with}$$
$$\Psi_A(0,0) = I; \qquad \Psi_A(t^-,\tau) = 0 \quad \textit{for all } \tau > t \geqslant 0, \tag{7}$$

which is defined explicitly as follows:

$$\Psi_A(t^-,0) = \Psi_{A0}(t,0)\left[I + \int_{0^-}^{t^-} \Psi_{A0}(0,\tau) \tilde{A}(\tau) \Psi_{A0}(\tau,0)\, d\tau \right] \tag{8a}$$

$$= \Psi_{A0}(t,0)\left[I + \int_{0^-}^{t^-} \Psi_{A0}(0,\tau) \tilde{A}'(\tau) \Psi_{A0}(\tau,0)\, d\tau \right]$$
$$+ \sum_{t_i \in TN(0,t)} \Psi_{A0}(0,t_i) A''(t_i) \Psi_{A0}(t_i^-,0) \quad \textit{for } t \geqslant 0 \tag{8b}$$

with $\tilde{A}(t) = A(t) - A_0 = \tilde{A}'(t) + A''(t)$ *and* $\Psi_{A0}(t,\tau) := e^{A_0(t-\tau)}$*, for all* t *and* τ*, is a* C_0*-semigroup of infinitesimal generator* A_0*, which is the evolution operator of* S1*, and*

$$\Psi_A(t^+,0) = \bigl(I + \mu(t)A''(t)\bigr)\Psi_A(t^-,0). \tag{9}$$

(iii) *Evolution Equations 3* (*EE3*) *from* S3.

$$x(t^-) = (S\varphi)(t^-) + \bigl(S^u u\bigr)(t^-), \tag{10a}$$

$$\begin{aligned} x(t^+) &= (S\varphi)(t^+) + \bigl(S^u u\bigr)(t^+) \\ &= \bigl(I + \mu(t)A''(t)\bigr)x(t^-) + \mu_{\mathrm{d}}(t)A''_{\mathrm{d}}(t)x\bigl(t^- - r(t)\bigr) + \mu_u(t)B(t)u''(t), \end{aligned} \tag{10b}$$

$$y(t^-) = (M\varphi)(t^-) + \bigl(M^u u\bigr)(t^-), \tag{10c}$$

$$\begin{aligned} y(t^+) &= (M\varphi)(t^+) + \bigl(M^u u\bigr)(t^+) \\ &= C(t)x(t^+) + D(t)\bigl(u'(t) + \mu_u(t)u''(t)\delta(0)\bigr), \end{aligned} \tag{10d}$$

where the linear operators $S \in \mathbf{L}(IC, L^n_{2\mathrm{e}})$*,* $S^u \in \mathbf{L}(L^m_2, L^n_{2\mathrm{e}})$*,* $M \in \mathbf{L}(IC, L^p_{2\mathrm{e}})$*,* $M^u \in \mathbf{L}(L^m_2, L^p_{2\mathrm{e}})$ *are defined point-wise as follows*:

$$\begin{aligned} &(S\varphi)(t^-) := T(t^-,0)x_0 + \int_{-\bar{r}}^{0^+} T(t^-,-\tau)A_{\mathrm{d}}\bigl(\tau + r(\tau)\bigr)\varphi(\tau)\,d\tau, \\ &\quad x_0 = x(0^+) - \varphi_{\mathrm{imp}}(0^+) = \varphi_0(0) + \tilde{\varphi}(0^+), \end{aligned} \tag{11a}$$

$$\begin{aligned} (S\varphi)(t^+) &:= T(t^+,0)x_0 + \int_{-\bar{r}}^{0^+} T(t^+,-\tau)A_{\mathrm{d}}\bigl(\tau + r(\tau)\bigr)\varphi(\tau)\,d\tau \\ &= \bigl(I + \mu(t)A''(t)\bigr)(S\varphi)(t^-) + \mu_{\mathrm{d}}(t)A''_{\mathrm{d}}(t)x\bigl(t^- - r(t)\bigr), \end{aligned} \tag{11b}$$

$$\begin{aligned} \bigl(S^u u\bigr)(t^-) &:= \int_{0^-}^{t^-} T(t^-,\tau)B(\tau)u(\tau)\,d\tau \\ &= \int_{0^-}^{t^-} T(t^-,\tau)B(\tau)u'(\tau)\,d\tau + \sum_{t_i \in TU} T(t^-,t_i)B(t_i)u''(t_i), \end{aligned} \tag{11c}$$

$$\begin{aligned} \bigl(S^u u\bigr)(t^+) &:= \int_{0^-}^{t^+} T(t^+,\tau)B(\tau)u(\tau)\,d\tau \\ &= \int_{0^-}^{t^-} T(t^+,\tau)B(\tau)u'(\tau)\,d\tau + \sum_{t_i \in TU} T(t^+,t_i)B(t_i)u''(t_i) + \mu_u(t)B(t)u''(t) \\ &= \bigl(I + \mu(t)A''(t)\bigr)(S_u u)(t^-) + \mu_u(t)B(t)u''(t), \end{aligned} \tag{11d}$$

$$\begin{aligned} &(M\varphi)(t^-) := C(t)(S\varphi)(t^-), \\ &(M\varphi)(t^+) := C(t)(S\varphi)(t^+), \end{aligned} \tag{11e}$$

$$\bigl(M^u u\bigr)(t^-) := C(t)\bigl(S^u u\bigr)(t^-) + D(t)u'(t^-), \tag{11f}$$

$$\begin{aligned}
\big(M^u u\big)(t^+) &:= C(t)\big(S^u u\big)(t^+) + D(t)\big(u'(t^+) + \mu_u(t)u''(t)\delta(0)\big),\\
&= C(t)\big[\big(I + \mu(t)A''(t)\big)(S_u u)(t^-) + \mu_u(t)B(t)u''(t)\big]\\
&\quad + D(t)\big(u'(t^+) + \mu_u(t)u''(t)\delta(0)\big)\\
&= C(t)\Bigg[\int_{0^-}^{t^-} T(t^+,\tau)B(\tau)u'(\tau)\,d\tau + \sum_{t_i\in TU} T(t^+,t_i)B(t_i)u''(t_i)\\
&\quad + \mu_u(t)B(t)u''(t) + \mu_u(t)B(t)u''(t)\Bigg] + D(t)\big(u'(t^+) + \mu_u(t)u''(t)\delta(0)\big),
\end{aligned} \tag{11g}$$

and $T(t,0)$ is an almost everywhere time-differentiable evolution operator that satisfies:

$$\begin{aligned}
\dot T(t^-,0) &= A'(t^-)T(t^-,0) + A_{\mathrm d}(t^-)T\big(t^- - r(t),0\big),\\
\dot T(t^+,0) &= A(t^+)T(t^+,0) + A_{\mathrm d}(t^+)T\big(t^+ - r(t),0\big)\\
&= \big(A'(t^+) + \mu(t)A''(t^+)\big)T(t^+,0) + \big(A'_{\mathrm d}(t^+) + \mu_{\mathrm d}(t)A''_{\mathrm d}(t)\big)T\big(t^+ - r(t),0\big)
\end{aligned} \tag{12a}$$

with $T(0,0)=I$, $T(t^-,\tau)=0$ for $\tau > t \geqslant 0$ so that fort $\geqslant 0$:

$$\begin{aligned}
T(t^-,0) &= \Psi_{A0}(t,0)\Bigg[I + \int_{0^-}^{t^-} \Psi_{A0}(0,\tau)\tilde A'(\tau)T(\tau,0)\,d\tau\\
&\quad + \int_{0^-}^{t^-} \Psi_{A0}(0,\tau)A'_{\mathrm d}(\tau)T\big(\tau - r(\tau),0\big)\mathbf{1}(t-\tau)\,d\tau\Bigg]\\
&\quad + \sum_{t_i\in TN(0,t)} \Psi_{A0}(t,t_i)A''(t_i)T(t_i^-,0)\\
&\quad + \sum_{t_i\in TN_{\mathrm d}(0,t)} \Psi_{A0}(t,t_i)A''_{\mathrm d}(t_i)T\big(t_i^- - r(t_i),0\big)
\end{aligned} \tag{12b}$$

$$\begin{aligned}
&= \Psi_A(t^-,0) + \int_{0^-}^{t^-} \Psi_A(t^-,\tau)A'_{\mathrm d}(\tau)T\big(\tau - r(\tau),0\big)\mathbf{1}(t-\tau)\,d\tau\\
&\quad + \sum_{t_i\in TN_{\mathrm d}(0,t)} \Psi_A(t^-,t_i)A''_{\mathrm d}(t_i)T\big(t_i^- - r(t_i),0\big)
\end{aligned} \tag{12c}$$

and

$$T(t^+,0) = \big(I + \mu(t)A''(t)\big)T(t^-,0) + \mu_{\mathrm d}(t)A''_{\mathrm d}(t)T\big(t^- - r(t),0\big). \tag{13}$$

Proof. *Part* (i). Note that $\Psi_{A0}(t,\tau) = e^{A_0(t-\tau)}$, $\forall t,\tau$ is a (C_0-semigroup) evolution operator for S1 with infinitesimal generator A_0 satisfying $\Psi_{A0}(t,t) = I$ and $\Psi_{A0}(t,\tau) = \Psi_{A0}(t-\tau) = \Psi_{A0}^{-1}(\tau,t) = e^{A_0(t-\tau)}$ with $f(t) = (A(t) - A_0)x(t) + A_{\mathrm d}(t)x(t - r(t)) + B(t)u(t)$ being a forcing function in (1a) with respect to the homogeneous system S1 for any $\varphi \in IC([-\bar r,0],\mathbf{R}^n)$; i.e. $x(t) \equiv z_{A0}(t)$ for any real bounded $x(0+) = z_{A0}(0) = x_0 = \varphi_0(0^+) + \tilde\varphi(0)$ if $f \equiv 0$ and $\varphi(t) = 0$,

$t \in [-\bar{r}, 0)$. Thus, the unique state and output solution trajectories of (1) satisfy the integral identities for $t > 0$:

$$x(t^-) = e^{A_0 t}\left[x_0^+ + \int_{0^-}^{t^-} e^{-A_0\tau} f(\tau)\,d\tau\right], \qquad y(t^-) = C(t)x(t^-) + D(t)u'(t), \tag{14a}$$

$$x(t^+) = \big(I + \mu(t)A''(t)\big)x(t^-) + \mu_{\mathrm{d}}(t)A''_{\mathrm{d}}(t)x\big(t^- - r(t)\big) + \mu_u(t)B(t)u''(t), \tag{14b}$$

$$y(t^+) = C(t)x(t^+) + D(t)\big(u'(t) + \mu_u(t)u''(t)\delta(0)\big), \tag{14c}$$

since $u(t^-) = u'(t^-)$ and $u(t^+) = u'(t^+) + \mu_u(t)u''(t)\delta(0)$. Equations (14a) are directly obtained by constructing the solution of (1a) via the homogeneous auxiliary system S1 and the use of (1c).

Part (ii). By comparing the auxiliary systems S2 and S1 with $\tilde{A}(t) = A(t) - A_0 = \tilde{A}'(t) + A'(t)$, the unique state-trajectory solution of S2 for $t > 0$ for any $z_0^+ = z_A(0^+) = \varphi_0(0^+) + \tilde{\varphi}(0)$ is given by

$$\begin{aligned} z_A(t^-) &= \Psi_A(t^-, 0)z_0^+ = \Psi_{A0}(t^-, 0)z_0^+ + \int_{0^-}^{t^-} \Psi_{A0}(t, \tau)\tilde{A}(\tau)z(\tau)\,d\tau \\ &= \Psi_{A0}(t^-, 0)\left(I + \int_{0^-}^{t^-} \Psi_{A0}(0, \tau)\tilde{A}(\tau)\Psi(\tau, 0)\,d\tau\right)z_0^+ \end{aligned} \tag{15}$$

since $\Psi_{A0}(t, \tau) = \Psi_{A0}(t, 0)\Psi_{A0}(\tau, 0) = e^{A_0(t-\tau)}$ for any t, τ. Direct calculation with (8) yields:

$$\dot{\Psi}_A(t^-, 0) = A_0\left\{\Psi_{A0}(t, 0)\left[I + \int_{0^-}^{t^-} \Psi_{A0}(0, \tau)\tilde{A}(\tau)\,d\tau\right]\right\} + \tilde{A}(t)\Psi(t^-, 0)$$

since $\dot{\Psi}_{A0}(t, \tau) = A_0\Psi_{A0}(t, \tau) = A_0 e^{A_0(t-\tau)}$, $\forall t, \tau$. On the other hand,

$$\begin{aligned} z_A(t^+) &= \Psi_A(t^+, 0)z_0^+ = \Psi_A(t^-, 0)z_0^+ + \left(\int_{t^-}^{t^+} A''(\tau)\delta(\tau - t)\Psi_A(\tau, 0)\,d\tau\right)z_0^+ \\ &= \big(I + \mu(t)A''(t)\big)\Psi_A(t^-, 0)z_0^+ = \big(I + \mu(t)A''(t)\big)z_A(t^-) \end{aligned} \tag{16}$$

which holds for any z_0^+ if and only if (9) holds. It has been proved that the evolution operator of S2 satisfies (8), (9). Now, since S3 has a forcing term $A_{\mathrm{d}}(t)z(t - r(t))$ with respect to S2, the unique state trajectory solution of S3 for any given $\varphi \in IC \in ([-\bar{r}, 0], \mathbf{R}^n)$, which is also the unique solution of the homogeneous (1a) for such an initial condition for all $t > 0$, is by construction:

$$\begin{aligned} z(t^-) &= \Psi_A(t^-, 0)z_0^+ + \int_{0^-}^{t^-} \Psi_A(t^-, \tau)A'_{\mathrm{d}}(\tau)z(\tau - r(\tau))\,d\tau \\ &\quad + \sum_{t_i \in TN_{\mathrm{d}}} \Psi_A(t^-, t_i)A''_{\mathrm{d}}(t_i)z\big(t_i^- - r(t_i)\big) \end{aligned} \tag{17a}$$

$$
\begin{aligned}
z(t^+) &= \Psi_A(t^+,0)z_0^+ + \int_{0^-}^{t^+} \Psi_A(t^+,\tau)A'_{\mathrm{d}}(\tau)z\big(\tau - r(\tau)\big)\,d\tau \\
&\quad + \sum_{t_i \in TN_{\mathrm{d}}} \Psi_A(t^+,t_i)A''_{\mathrm{d}}(t_i)z\big(t_i^+ - r(t_i)\big) && \text{(17b)}\\
&= \big(I + \mu(t)A''(t)\big)z(t^-) + \mu_{\mathrm{d}}(t)A''_{\mathrm{d}}(t)z\big(\tau - r(\tau)\big)\,d\tau. && \text{(17c)}
\end{aligned}
$$

Now, it follows that the homogeneous solution of S3 (i.e., that of (1a) for $u \equiv 0$) for t $\geqslant 0$ is

$$
\begin{aligned}
z(t^-) &= (S_A\varphi)(t^-) + (S'_A z_{[t]})(t^-),\\
z(t^+) &= (S_A\varphi)(t^+) + (S'_A z_{[t]})(t^+)
\end{aligned}
$$

for any given $\varphi \in IC \in ([-\bar{r},0], \mathbf{R}^n)$ while that of S2 is

$$
\begin{aligned}
z_A(t^-) &= (S_A z_0^+)(t^-),\\
z_A(t^+) &= (S_A z_0^+)(t^+)
\end{aligned}
$$

with $z_0^+ = z_A(0^+) = \varphi_0(0) + \tilde{\varphi}(0^+)$. Then, the unique state-trajectory solution of S from (1a) is then uniquely given by construction by (5a), (5b). Combining (1b) with (5a), (5b) yields directly the output-trajectory solution (5c), (5d) on $[0,\infty)$ with the operator definitions (6e), (6g) and the appropriate replacements $t^- \to t^+$.

Part (iii). If the state-trajectory solution satisfies (10a) with $z(t) \equiv \varphi(t)$ for any given $\varphi \in IC([-\bar{r},0), \mathbf{R}^n)$ satisfying the corresponding equations (11). Then, for any $z(0^+) = z_0^+ = \varphi_0(0) + \tilde{\varphi}(0^+)$ and all $t > 0$,

$$
\begin{aligned}
\dot{z}(t^-) &= \dot{T}(t^-,0)z_0^+ + \int_{-\bar{r}}^{0^-} \dot{T}(t^-,-\tau)A_{\mathrm{d}}\big(\tau + r(\tau)\big)\varphi(\tau)\,d\tau \\
&= A(t^-)z(t^-) + A_{\mathrm{d}}(t^-)z(t^-) && \text{(18)}
\end{aligned}
$$

by using the definition of $T(t^-,0)$ in (12) and $T(t,\tau) = 0$ for $\tau > t$. Similarly, the definition of $T(t^+,0)$ and a similar derivation as that of (18) yields:

$$
\begin{aligned}
\dot{z}(t^+) &= A(t^+)z(t^+) + A_{\mathrm{d}}(t^+)z\big(t^+ - r(t)\big), && \text{(19)}\\
z(t^+) &= (S\varphi)(t^+) = z(t^-) + \int_{t^-}^{t^+} \dot{z}(\tau)d\,\tau \\
&= \big(I + \mu(t)A''(t)\big)(S\varphi)(t^-) + \mu_{\mathrm{d}}(t)A''_{\mathrm{d}}(t)(S\varphi)\big(t^- - r(t)\big).
\end{aligned}
$$

Then, the state trajectory solution of S3, and thus that of S, Eq. (1a), is satisfied on $(0,\infty)$ for $u \equiv 0$ by (10a), (10b) via the corresponding definitions of the operators $(S\varphi)(t^-)$, $(S\varphi)(t^+)$, $T(t^-,0)$ and $T(t^+,0)$ via (11), (12) provided that $z(t) \equiv \varphi(t)$, $t \in [-\bar{r},0]$ for all $\varphi \in IC([-\bar{r},0], \mathbf{R}^n)$.

The forced state-trajectory solution (10a)–(10b) follows by direct construction from the homogeneous solution of S3. The output trajectory solution (10c)–(10d) follows directly from (1b) via (10a)–(10b) by replacing the operators $S \in \mathbf{L}(IC, L_{2e}^n)$ and $S^u \in \mathbf{L}(L_2^m, L_{2e}^n)$ by $M \in \mathbf{L}(IC, L_{2e}^p)$ and $S^u \in \mathbf{L}(L_2^m, L_{2e}^p)$ defined in (11), respectively. $\square$

4. Stability

The following result is concerned with sufficient conditions for global exponential stability (GES) of the system S (via obtaining related properties for the auxiliary systems S1, S2), what implies global asymptotic stability (GAS) in the sense that the state trajectory of the unforced system vanishes exponentially, respectively, asymptotically with time for any $\varphi \in IC([-\bar{r}, 0], \mathbf{R}^n)$ for any bounded $z(0^+) \equiv \varphi(0^+) = \varphi_0(0) + \tilde{\varphi}(0^+)$. Such conditions also imply the global stability (GE) of the forced system S and that its output $y \in L_2^p$, for any input $u \in L_2^m$, and that $y \in L_2^p \cap L_\infty^p$ if $u \in L_2^m \cap L_\infty^m$ (i.e. when u is square-integrable but not impulsive). The conditions concerning the "smallness" of the absolute values of certain parameters referred to in the result statement are then made explicit in the corresponding parts of the proof.

4.1. Main stability result

Theorem 2. *The following items hold*:

(i) *Let A_0 be a stability matrix with fundamental matrix of the associated differential system being the evolution C_0-semigroup $\Psi_{A0}(t, \tau)$ satisfying $\|\Psi_{A0}(t, \tau)\| \leqslant k_0 e^{-\rho_0(t-\tau)}$ for all t, τ, some real* (*norm-dependent*) *constants $k_0 \geqslant 1$ and all real constants $\rho_0 \in (0, \rho^*)$ where $(-\rho^*) < 0$ is the stability abscissa of A_0* (*i.e., the largest real part of its eigenvalues. If the eigenvalue of largest real part is simple then $\rho \in (0, \rho^*]$*). *Thus* S2 *is GES if $0 \notin TN$ and*

$$\rho_0 > \underline{\rho_0} := \sup_{t \in \mathbf{R}_0^+} \left(\frac{k_0}{t} \left[\int_{0^-}^{t^-} \| \tilde{A}'(\tau) \| \, d\tau + \sum_{t_i \in TN(0,t)} \| A'(t_i) \| \right] \right) \tag{20}$$

for any (*vector-induced*) *matrix norm $\|(\cdot)\|$.*

(ii) *Assume that $0 \notin TN$ and that there exist finite nonnegative real constants*

$$a := \operatorname*{ess\,sup}_{t \in \mathbf{R}_0^+} \left(\| \tilde{A}(t) \| \right), b \geqslant \sup_{t \in \mathbf{R}_0^+} \left(\frac{1}{t} \sum_{t_i \in TN(0,t)} \| A''(t_i) \| \right).$$

Then, S2 *is GES if $\rho_o > k_0(a + b)$.*

(iii) *Assume that $A'(t)$ has uniformly bounded entries on $[0, \infty)$ and eigenvalues satisfying* $\mathrm{Re}[\lambda_i(A'(t)) \leqslant -\sigma < 0]$, $\forall t \geqslant 0$, *and that it exist positive real constants $T_1 > 0$, $T_2 > 0$ and $T > 0$ such that*:

- *Any two consecutive $t_i, t_{i+1} \in TN$ satisfy $t_{i+1} - t_i \geqslant T_1$; i.e. $TN(t, t + T_1)$ contains at most a $t_i \in TN$* (*otherwise, it is empty*), $\forall t \geqslant 0$.
- *Any two consecutive $t_i, t_{i+1} \in TD$ satisfy $t_{i+1} - t_i \geqslant T_2$; i.e. $TD(t, t + T_2)$ contains at most a $t_i \in TD$* (*otherwise, it is empty*), $\forall t \geqslant 0$.
- *Real constants α_0 and α_1 exist such that*

$$\int_{t^-}^{t^- + T} \| \dot{A}'(\tau) \| \, d\tau \leqslant \alpha_1 T + \alpha_0, \quad \forall t \geqslant 0.$$

Then S2 *is GES for all real constants $\alpha_1 \in [0, \alpha_1^*)$ and $\varepsilon \in [0, \varepsilon^*)$, some sufficiently small $\alpha_0, \alpha_1^* \in \mathbf{R}^+$ and $\varepsilon \in \mathbf{R}^+$ where $\varepsilon := \max(\varepsilon_1 + \varepsilon_2, \varepsilon_3)$ with*

$$\begin{aligned}
\varepsilon_1 &:= \sup_{t\in S_1(t,t+T)}\left(\ln\frac{z_A^T(t^+)P(t^-)z_A(t^+)}{z_A^T(t^-)P(t^-)z_A(t^-)}\right)\\
&\leqslant \varepsilon_1^* := \sup_{t\in S_1(t,t+T)}\left(\ln\frac{\lambda_{\min}[(I+\mu(t)A''^T(t))P(t^-)(I+\mu(t)A''(t))]}{\lambda_{\max}(P(t^-))}\right),\\
\varepsilon_2 &:= \sup_{t\in S_2(t,t+T)}\left(\ln\frac{z_A^T(t)P(t^+)z_A(t)}{z_A^T(t)P(t^-)z_A(t)}\right)\leqslant \varepsilon_2^* := \sup_{t\in S_2(t,t+T)}\left(\ln\frac{\lambda_{\min}(P(t^+))}{\lambda_{\max}(P(t^-))}\right),\\
\varepsilon_3 &:= \sup_{t\in S_3(t,t+T)}\left(\ln\frac{z_A^T(t^+)P(t^+)z_A(t^+)}{z_A^T(t^-)P(t^-)z_A(t^-)}\right)\\
&\leqslant \varepsilon_3^* := \sup_{t\in S_3(t,t+T)}\left(\ln\frac{\lambda_{\min}[(I+\mu(t)A''^T(t))P(t^+)(I+\mu(t)A''(t))]}{\lambda_{\max}(P(t^-))}\right), \qquad (21)
\end{aligned}$$

where $S_i(t,t+T)$ $(i=1,2,3)$ are the empty or nonempty sets of time instants of impulses, jump discontinuities or combined impulses and jump discontinuities in the real interval $(t,t+T]$ defined as:

$$\begin{aligned}
S_1(t,t+T) &:= TN(t,t+T)\cap\overline{TD}(t,t+T),\\
S_2(t,t+T) &:= TD(t,t+T)\cap\overline{TN}(t,t+T),\\
S_3(t,t+T) &:= TN(t,t+T)\cap TD(t,t+T) \qquad (22)
\end{aligned}$$

and $P(t)=P^T(t)>0$ is a real matrix function $P:\mathbf{R}_0^+\to\mathbf{R}^{n\times n}$ that satisfies a Lyapunov matrix equation for some $q_0\in\mathbf{R}^+$:

$$\begin{aligned}
&A'^T(t)P(t)+P(t)A'(t) = -q_0 I \quad \text{for all } t\notin TD,\\
&A'^T(t)P(t^+)+P(t^+)A'(t) = A'^T(t^-)P(t^-)+P(t^-)A'(t^-)\\
&\qquad = -q_0 I \quad \text{for } t\in TD. \qquad (23)
\end{aligned}$$

(iv) *If the growing rate condition on $\|A'(t)\|$ in* (iii) *is replaced with*

$$\int_{t^-}^{t^-+T}\|\dot A'(\tau)\|\,d\tau\leqslant\alpha_1T+\alpha_0+\alpha_0',\quad \forall t\geqslant 0,\quad \text{where}$$

$$\alpha_0' := \sup_{t\in\mathbf{R}_0^+}\left(\sum_{\tau\in TD(t,t+T)}\left(\|A'(\tau^+)-A'(\tau^-)\|\right)\right),$$

then S2 *is GES for all real $\alpha_1\in[0,\bar\alpha_1^*)$, some $\bar\alpha_1^*\in\mathbf{R}^+$ if $\varepsilon\in[0,\varepsilon^*)$, for some sufficiently small $(\alpha_0+\alpha_0')$, $\bar\alpha_1^*\in\mathbf{R}^+$ and $\varepsilon^*\in\mathbf{R}^+$ where $\varepsilon,\varepsilon_i$ $(i=1,2,3)$ are defined in* (iii).

(v) *Items* (iii), (iv) *still hold under the respective modified constraints*:

$$\int_{t^-}^{t^-+T}\|\dot A'(\tau)\|^2\,d\tau\leqslant\alpha_1^2T+\bar\alpha_0,\quad \forall t\geqslant 0, \qquad (24)$$

$$\int_{t^-}^{t^-+T}\|\dot A'(\tau)\|^2\,d\tau\leqslant\alpha_1^2T+\bar\alpha_0+\bar\alpha_0',\quad \forall t\geqslant 0, \qquad (25)$$

and similar remaining conditions, where

$$\bar{\alpha}_0' := \sup_{t\in\mathbf{R}_0^+}\left(\sum_{\tau\in TD(t,t+T)} \left\|A'(\tau^+)-A'(\tau^-)\right\|^2\right),$$

provided that $\alpha_1 \in [0, \bar{\alpha}_1^*)$ (*respectively* $\alpha_1 \in [0, \bar{\alpha}_1^{*\prime})$ *with* $\bar{\alpha}_1^*$ (*respectively* $\bar{\alpha}_1^{*\prime}$) *and* α_0 (*respectively* $\bar{\alpha}_0 + \bar{\alpha}_0'$) *sufficiently small.*

(vi) *Assume that* $A'(t)$ *is locally integrable for all* $t \geqslant 0$ *and that for some integers* n_0 ($1 \leqslant n_0 \leqslant n$) *and* n_0', *there is a* $n_0 \times n_0'$ *matrix function* $N(t)$ *such that the Lyapunov matrix equation*

$$A'^T(t)P(t) + P(t)A'(t) = -N(t)N^T(t) - q_0 I \tag{26}$$

holds for all $t \in \mathbf{R}_0^+$, *some real square* n-*matrix* $P = P^T > 0$ *and some* $q_0 \in \mathbf{R}_0^+$. *Then,* S2 *is GES if* $A''(t_i)$ ($t_i \in TN$) *is uniformly bounded, the pair* $[A'(t), N^T(t)]$ *is uniformly completely observable if* $q_0 = 0$ (*not required if* $q_0 > 0$) *and*

$$\prod_{t_i\in TN(t,t+T)}\left[\frac{z_A^T(t_i^-)(I + A''^T(t_i))P(I + A''(t_i))z_A(t_i^-)}{z_A^T(t_i^-)Pz_A(t_i^-)}\right] < \frac{\lambda_{\max}(P)}{\alpha}, \tag{27}$$

where αI *is a lower-bound matrix of the observability Grammian* $\mathrm{G}(t, t+T)$ *of* $[A'(t), N^T(t)]$ (*i.e.* $\mathrm{G}(t, t+T) - \alpha I \geqslant 0$).

(vii) *Assume a real sequence* $S_t \equiv \{t_i\}_1^\infty$ *defined by* $S_t := \{t_i \in \mathbf{R}_0^+\colon\ t_{i+1} - t_i \geqslant T_* > 0\}$ *for some real fixed* $T_* > 0$. *Thus,* (vi) *still holds if the subsequent Lyapunov matrix inequality is satisfied*:

$$A'^T(t)P(t) + P(t)A'(t) \leqslant -N(t)N^T(t) - q_0(t_i)I, \quad \forall t \in [t_i, t_{i+1})$$

with the real sequence $\{q_0(t_i)\}_1^\infty$ *satisfying* $q_0(t_i) \geqslant q_0 > 0$ *for all real interval* $[t_i, t_{i+1})$, $t_i \in S_t$, *provided that* (27) *holds where* $[A'(t), N^T(t)]$ *is not uniformly completely observable.*

(viii) *The auxiliary system* S3, *and then the unforced* S (1a), *is GES for all* $\varphi \in IC([-\bar{r}, 0], \mathbf{R}^n)$ *if* S2 *is GES satisfying all the conditions of* (iii) [*or, alternatively, those of* (vi) *or* (v)], *for some positive real constant* T, *such that the Lyapunov matrix equations hold with* q_0 *satisfying*

$$\sup_{t\notin TD}\left(\left\|A_{\mathrm{d}}^T\left(h^{-1}(t)\right)A_{\mathrm{d}}(h^{-1}(t))\right\|\right) < q_0 < \frac{4\sigma_A^2}{\alpha_A^4}$$

and, furthermore, $\dot{r}(t) \leqslant \gamma < 1$ *for all* $t \in CD$, *and*

(a) $\|A_{\mathrm{d}}'^T(h^{-1}(t))A_{\mathrm{d}}'(h^{-1}(t))\|$ *is finite for all* $t \in C\overline{D} \cup DD$ *where* $h(t) := t - r(t)$, $C\overline{D}$ *is the set of zero measure where the delay function* $r : \mathbf{R}_0^+ \to \mathbf{R}_0^+$ *is continuous non-differentiable with respect to time, DD is the set of zero measure where it is discontinuous*;

(b)

$$\int_{1/h(t_1^-)}^{1/h(t_2^-)} \left\|A_{\mathrm{d}}'(\tau)\right\|^2\left(1 - \dot{r}(\tau)\right)d\tau \leqslant \gamma'(t_2 - t_1) + \gamma_0'(t_1, t_2) \tag{28}$$

for all $t_2 \geqslant t_1 > 0$ *such that* $(C\overline{D} \cup DD) \cap [t_1, t_2) = \emptyset$ *for some nonnegative finite real constants* $\gamma'(t_2 - t_1)$ *and* $\gamma_0'(t_1, t_2)$. *If* $[t_1, t_2) \cap TN = \emptyset$ *then* $\gamma_0'(t_1, t_2) = 0$;

(c)

$$e^{-qT}\left(\prod_{t_i\in D(t,t+T)}\left[\lambda(t_i)\right]\right)<1; \qquad \lambda(t_i):=\frac{V'(t_i^+)}{V'(t_i^-)}, \tag{29}$$

where

$$q:=q_0\left(1-\frac{\alpha_A^4 q_0}{4\sigma_A^2}\right)-\sup_{t\notin D}\left(\left\|A_{\mathrm{d}}'^T\left(h^{-1}(t)\right)A_{\mathrm{d}}'\left(h^{-1}(t)\right)\right\|\right)>0$$

for some real constant $q_0\geqslant \underline{q_0}>0$ *with sets* $D(t,t+T):=TN(t,t+T)\cup TN_{\mathrm{d}}(t,t+T)\cup TD(t,t+T)\cup DD(t,t+T)$ *and* $DD(t,t+T):=DD(t)\cap DD(t,t+T)$ *of discontinuities, and*

$$V'(t):=z^T(t)P(t)z(t)+\int_{t^- - r(t)}^{t^-} z^T(\tau)A_{\mathrm{d}}^T\left(h^{-1}(\tau)\right)A_{\mathrm{d}}\left(h^{-1}(\tau)\right)z(\tau)\,d\tau. \tag{30}$$

(ix) *Assume that there exist a matrix function* $N(t)$ *and* $q_0\in\mathbf{R}_0^+$ *satisfying* $q_0<4\sigma_A^2/\alpha_A^4$ *such that*

$$A'^T(t)P(t)+P(t)A'(t)\leqslant -q_0 I-N(t)N^T(t) \tag{31}$$

with $A'(t)$ *being locally integrable and* $A''(t)$. *Assume also that the matrix function* $A_{\mathrm{d}}':\mathbf{R}_0^+\to\mathbf{R}^{n\times n}$ *satisfies the same conditions as in* (viii) *and* (28) *is fulfilled. Then* S3, *and then the unforced* (1a) *of* S, *is GES for all* $\varphi\in IC([-\bar r,0],\mathbf{R}^n)$.

(x) *Assume that* $A'(t)$ *is locally integrable,* $A''(t)$ *is uniformly bounded and for some positive real constants* α,β $(\beta\geqslant\alpha)$, *and, in addition,*

$$\beta I\geqslant \int_{0^-}^{t^-} T^T(t,\tau)N(\tau)N^T(\tau)T(t,\tau)\,d\tau\geqslant \alpha I,\quad \forall t\geqslant 0, \tag{32}$$

if (26) *holds with* $q_0=0$. *Then* S3, *and then the unforced* S, *is GES for all* $\varphi\in IC([-\bar r,0],\mathbf{R}^n)$ *while* S *is GE for all* $\varphi\in IC([-\bar r,0],\mathbf{R}^n)$ *and* $u\in L_2^m$ *and, furthermore,*

$$\frac{\alpha\vartheta_t'(0)}{\lambda_{\max}(P)}>\varepsilon_0,\qquad \left(1-\frac{\alpha\vartheta_T'(t)}{\lambda_{\max}(P)}\right)\prod_{t_i\in D(t,t+T)}\left[\lambda(t_i)\right]<1-\varepsilon_0, \tag{33}$$

with

$$0<\vartheta_T'(t)\leqslant e^{-q_0T},\quad \lambda(t_i):=\frac{z^T(t_i^+)Pz(t_i^+)}{z^T(t_i^-)Pz(t_i^-)},$$

$$0<\varepsilon_0\leqslant \alpha k_\varphi \sup_{t\in\mathbf{R}_0^+}\left(\frac{\|T(t,0)\|_2^2\vartheta_T(t)}{\lambda_{\max}(P)}\right)<1 \tag{34}$$

for some real constant k_φ, *dependent on the initial conditions, and* $\forall t\geqslant 0$.

(xi) *Assume that* $A'(t)$ *is locally integrable,* $A''(t)$ *is uniformly bounded and* $[A'(t),N^T(t)]$ *is uniformly completely observable*; *i.e. for some positive real constants* α,β $(\beta\geqslant\alpha)$ *such that*

$$\beta I\geqslant \int_{t^-}^{t^-+T} T^T(t,\tau)N(\tau)N^T(\tau)T(t,\tau)\,d\tau\geqslant \alpha I,\quad \forall t\geqslant 0. \tag{35}$$

Then S3*, and then the unforced* S*, is GES for all* $\varphi \in IC([-\bar{r}, 0], \mathbf{R}^n)$ *while* S *is GE for all* $\varphi \in IC([-\bar{r}, 0], \mathbf{R}^n)$ *and* $u \in L_2^m$ *if the Lyapunov equation* (26) *and* (33)*,* (34) *hold with* $q_0 = 0$*.*

4.2. Partial proof of Theorem 2

In order not to overlength the paper, only items (i), based on Gronwall lemma and (viii), based on Lyapunov theory are proved. The remaining items may be proved with very close techniques to those ones so that their proofs are omitted.

(i) One gets directly from the differential system defining S2:

$$\left\| z_A(t^+) \right\| \leqslant k_0 \left[e^{-\rho_0 t} \left\| z_A(0^-) \right\| + \int_{0^-}^{t^+} e^{-\rho_0 \tau} \left\| \tilde{A}(\tau) \right\| \left\| z_A(\tau) \right\| d\tau \right] \tag{36}$$

for all $t \geqslant 0$ and any vector norm $\|m\|$ and associated induced matrix norm $\|M\|$, $\forall m \in \mathbf{R}^n$ and $M \in \mathbf{R}^{n \times n}$, since $\|\Psi_{A0}(t, \tau)\| \leqslant k_0 e^{-\rho_0 (t-\tau)}$ with some real constants $k_0 \geqslant 1$ (norm-dependent) and $\rho_0 > 0$ (since A_0 is a stability matrix). Using Gronwall's lemma [11] in (36) it follows that

$$\left\| z_A(t^+) \right\| \leqslant k_0 \left[\left\| z_A(0^-) \right\| e^{-(\rho_0 t - \int_{0^-}^{t^+} \|\tilde{A}(\tau)\| v(\tau)\, d\tau)} \right] \tag{37}$$

for all $t \geqslant 0$. Since

$$\int_{0^-}^{t^+} \left\| \tilde{A}(\tau) \right\| d\tau \leqslant \int_{0^-}^{t^+} \left\| \tilde{A}'(\tau) \right\| d\tau + \sum_{t_i \in TN} \left\| A''(t_i) \right\|,$$

S2 is GES for all $\varphi \in IC([-\bar{r}, 0], \mathbf{R}^n)$ if $\rho_0 > \underline{\rho_0}$, which is finite since $0 \notin TN$ what implies $A''(0) = 0$.

(viii) Consider the Lyapunov function candidate (30). It is first proved that $\int_{t^- - r(t)}^{t^-} z^T(\tau) K(\tau) \times z(\tau)\, d\tau$ does not diverge faster than $r(t)(\sup_{0 \leqslant \tau \leqslant t}(\|z(\tau)\|_2^2))$, where $K(t) := A_{\mathrm{d}}'^T(h^{-1}(t)) \times A_{\mathrm{d}}'(h^{-1}(t))$. Direct calculus yields:

$$\begin{aligned} \Delta V(t) := V'(t) - V(t) &= \int_{t-r(t)}^{t} \left\| A_{\mathrm{d}}'\big(h^{-1}(\tau)\big) z(\tau) \right\|_2^2 d\tau \\ &\leqslant \operatorname*{ess\,sup}_{t-r(t) \leqslant \tau \leqslant t} \left(\left\| z(\tau) \right\|_2^2 \right) \int_{\frac{1}{t^- - r(t)}}^{\frac{1}{t^-}} \int_{t-r(t)}^{t} \left\| A_{\mathrm{d}}'\big(h^{-1}(\tau)\big) \right\|_2^2 \dot{h}(\tau)\, d\tau \\ &\leqslant \mu_{\mathrm{a}} r(t) \operatorname*{ess}_{t-r(t) \leqslant \tau \leqslant t} \left(\left\| z(\tau) \right\|_2^2 \right) \end{aligned} \tag{38}$$

for all $t \notin C\overline{D} \cup DD$ for any vector and associate (induced) matrix norm and some $\mu_{\mathrm{a}} \in \mathbf{R}_0^+$. Since $\|A_{\mathrm{d}}(1/t)\|$ is bounded if $t \in C\overline{D} \cup DD$, it exists $\mu_{\mathrm{b}} \in \mathbf{R}^+$ such that

$$\Delta V(t) \leqslant \mu_{\mathrm{b}} r(t) \operatorname*{ess\,sup}_{t-r(t) \leqslant \tau \leqslant t} \left(\left\| z(\tau) \right\|_2^2 \right).$$

Also,

$$\dot{V}'(t) = z^T(t)\big[A'^T(t)P(t) + P(t)A'(t) + K(t)\big]z(t) + 2z^T(t)P(t)A'_{\mathrm{d}}(t)z\big(h(t)\big) - \big(1 - r'(t)\big)z^T\big(h(t)\big)K\big(h(t)\big)z\big(h(t)\big) - z^T(t)\dot{P}(t)z(t) \tag{39}$$

for all $t \notin C\overline{D}$ where $r(t) := 1 - \dot{r}(t)$. Assume that $q \in \mathbf{R}^+$ exists such that

$$q_0 \geqslant q + \sup_{t \notin C\overline{D}} \big[A'^T_{\mathrm{d}}\big(h^{-1}(t)\big)A'_{\mathrm{d}}\big(h^{-1}(t)\big)\big] + \frac{q_0^2 \alpha_A^4}{4\sigma_A^2} \tag{40}$$

with q_0, σ_A and α_A being real constants obtained in the proof of (iii). Direct calculus guarantees (40) if for all

$$t \notin D_{\mathrm{e}},\ 0 < q_0 < \frac{4\sigma_A^2}{\alpha_A^4}, \quad \lambda_{\max}\big[A'^T_{\mathrm{d}}\big(h^{-1}(t)\big)A'_{\mathrm{d}}\big(h^{-1}(t)\big)\big] < q_0\left(1 - \frac{q_0\alpha_A^4}{4\sigma_A^2}\right)$$

so that

$$A'^T(t)P(t) + P(t)A'(t) + \dot{P}(t) = -q_0 I \leqslant -\big(qI + P^2(t) + A'^T_{\mathrm{d}}\big(h^{-1}(t)\big)A'_{\mathrm{d}}\big(h^{-1}(t)\big)\big) \tag{41}$$

for $t \notin D_{\mathrm{e}}$ provided that (40) holds. The substitution of (41) into (39) and the use of (iii) yields

$$\begin{aligned} \dot{V}(t) &\leqslant -\big(q_0\beta_2^{-1} - \beta\beta_1^{-1}\big\|\dot{A}'(t)\big\|_2^2\big)z^T(t)P(t)z(t) \\ &\quad - \left\| \frac{1}{\sqrt{1 - r'(t)}}P(t)z(t) - \sqrt{1 - r'(t)}A_{\mathrm{d}}(t)z\big(h(t)\big)\right\|_2^2 \\ &\leqslant -\big(q_0\beta_2^{-1} - \beta\beta_1^{-1}\big\|\dot{A}'(t)\big\|_2\big)z^T(t)P(t)z(t) \leqslant 0, \end{aligned} \tag{42}$$

where $\|\cdot\|_2$ denotes the ℓ_2 vector and associated induced matrix norms. Thus, $\dot{V}'(t) \leqslant -qV'(t)$ for $\forall t \notin D_{\mathrm{e}}$ since $V'(t) \geqslant z^T(t)P(t)z(t)$. If $t_i \in C\overline{D}$ and $C\overline{D} \cap (TN \cup TN_{\mathrm{d}}) = \emptyset$ then $V'(t_i^+) = V'(t_i^-)$ (so that $C\overline{D}$ is irrelevant for stability analysis purposes). If $t_i \in (TN \cup TD \cup DD \cup TN_{\mathrm{d}})$ then $V'(t_i^+) \neq V'(t_i^-)$, in general, through respective jump discontinuities $\dot{z}(t_i^+) \neq \dot{z}(t_i^-)$; $P(t_i^+) \neq P(t_i^-)$, and

$$\Delta V(t_i^+) = V'(t_i) - z^T(t_i)P(t_i)z(t_i) = \int_{t_i - r(t_i)}^{t_i} \big\|A'_{\mathrm{d}}\big(h^{-1}(\tau)\big)z(\tau)\big\|_2^2\, d\tau. \tag{43}$$

The various discontinuities at $V'(t_i^+)$ may be combined since the above discontinuity sets are not required to be disjoint. Thus, if

$$e^{-qT}\left[\prod_{t_i \in D(t.t+T)} \big[\lambda(t_i)\big]\right] < 1$$

and taking into account (42), (43),

$$V'(t+T) \leqslant e^{-qT} \prod_{t_i \in D(t,t+T)} \big[\lambda(t_i)\big]V'(t_i) \leqslant V'(t^-), \quad \forall t \geqslant 0, \tag{44}$$

with $D(t+T) := \{\tau \in D\colon t \leqslant \tau < t+T\}$ and $\lambda(t_i) := V'(t_i^+)/V'(t_i^-)$. It has been proved that S2 is GES if (iii) holds. The proof is quite similar if S2 satisfies (iv) or (v).

5. Compactness of defining the state/output trajectory solutions

Note that the Banach space $U \subset L_2^m$ is also a real Hilbert space endowed with the (semi)norm of the inner product $\|\cdot\|_{L_2^m}$ defined by $\|u\|_{L_2^m} := \langle u, u\rangle_{L_2^m}^{1/2}$, $\forall u \in U$. The subscript for the space L_2^m for inner products and associate endowed norms are omitted in the sequel when no confusion is expected. Let $\{\phi_i^{(m)}\}_1^\infty$ and $\{\theta_i^{(m)*}\}_1^\infty = \{\theta_i^{(m)T}\}_1^\infty$ be an orthonormal basis and its reciprocal one for the L_2^m-space (this notation for orthonormal basis is adopted independently of the dimension m which is easily elucidated depending on context) so that $\langle \phi_i^{(m)}, \phi_j^{(m)}\rangle = \delta_{ij}$; $\langle \theta_i^{(m)}, \theta_j^{(m)}\rangle = \delta_{ij}$; $\langle \phi_i^{(m)}, \theta_j^{(m)}\rangle = \delta_{ij}$; $i, j = 1, 2, \ldots, \infty$ with δ_{ij} being the Kronecker delta. Then, the evolution operator $T(t, \tau)$ has a representation

$$T(t,\tau) = \sum_{k=1}^{\infty} \psi_\ell(t)\theta_k^{(m)^T}(\tau) = \sum_{k=1}^{\infty} \int_0^t T(t,\tau)\phi_k(\tau')\theta_k^{(m)^T}(\tau)\,d\tau', \tag{45}$$

where

$$\psi_k(t) := \int_{-\infty}^{\infty} T(t,\tau)\phi_k^{(m)}(\tau)\,d\tau = \int_0^t T(t,\tau)\phi_k^{(m)}(\tau)\,d\tau,$$

since $T(t,\tau) = 0$ for $t \geqslant \tau$, is a n-matrix function which is the image of ϕ_k via the operator T. The state and output trajectories are:

$$x(t) = (S\varphi)(t) + \big(S^u u\big)(t), \qquad y(t) = (M\varphi)(t) + \big(M^u u\big)(t) \tag{46}$$

for all $t \in \mathbf{R}_0^+$ from Theorem 1 so that for zero initial conditions:

$$x(t^-)]_{\varphi\equiv 0} = \big(S^u u\big)(t^-) = \int_0^{t^-} H_x(t^-,\tau)u(\tau)\,d\tau,$$

$$y(t^-)]_{\varphi\equiv 0} = \big(M^u u\big)(t^-) = \int_0^{t^-} H(t^-,\tau)u(\tau)\,d\tau \tag{47}$$

with similar expressions for $x(t^+)$ and $y(t^+)$ with the appropriate replacements $t^- \to t^+$. The Kernels of the operators $S^u \in \mathbf{L}(U,X)$ and $M^u \in \mathbf{L}(U,Y)$ are, respectively $H_x : [0,\infty) \to \mathbf{L}(U,X)$ and $H : [0,\infty) \to \mathbf{L}(U,Y)$, respectively. Let now be $\{\phi_i\}_1^\infty \equiv \{\phi_i^{(1)}\}_1^\infty$ an orthonormal basis of the real (scalar) space L_2. It follows that $\{\phi_{i_j}^{(\ell)}\}_1^\infty \equiv \{e_j^{(\ell)}\phi_i\}_1^\infty$ (i_j, $j = 1, 2, \ldots, \ell$) is an orthonormal basis of L_2^ℓ (any $\ell \in \mathbf{Z}^+$) for unity Euclidean vectors $e_j^{(\ell)} \in R^\ell$; i.e. its jth component is unity while the remaining ones are zero for $j = 1, 2, \ldots, \ell$ so that the inner product satisfies $\langle e_j^{(m)}\phi_i, e_\ell^{(m)}\theta_k\rangle = \delta_{j\ell}\delta_{ik}$. Then,

$$\varphi(t) = \sum_{k=1}^{\infty}\sum_{j=1}^{n} \alpha_{\varphi kj} e_j^{(n)}\phi_k(t) \quad \text{with } x_0 = \varphi(0) = \sum_{k=1}^{\infty}\sum_{j=1}^{n} \alpha_{\varphi kj} e_j^{(n)}\phi_k(0)$$
$$\text{for all } \varphi \in IC\big([-\bar{r}, 0], \mathbf{R}^n\big)$$

admitting a component-wise representation

$$\varphi_j(t)=\sum_{k=1}^{\infty}\alpha_{\varphi kj}\phi_k(t),\quad \text{with } x_j(0)=\varphi_j(0)=\sum_{k=1}^{\infty}\alpha_{\varphi kj}\phi_k(0),$$

where the $\alpha_{\varphi kj}$ $(j=1,2,\ldots,n;\ k=0,1,\ldots)$ are the (real constant) components of φ_j in the basis $\{\phi_k\}_1^{\infty}$. An artifice is now used to represent (possibly impulsive) inputs $u\in L_2^m$ of the class of the above sections. Such an artifice consist of introducing time-varying components which are essentially discontinuous (and not constant) only at discontinuity points so that

$$u(t^-)=u'(t^-)=\sum_{k=1}^{\infty}\sum_{j=1}^{n}\alpha_{kj}^{-}e_j^{(m)}\phi_k(t),$$

and

$$\begin{aligned}u(t^+)=u'(t^+)+u''(t)&=\sum_{k=1}^{\infty}\sum_{j=1}^{n}\alpha_{kj}e_j^{(m)}\phi_k(t)\\&=\sum_{k=1}^{\infty}\sum_{j=1}^{n}\bigl(\alpha_{kj}^{-}+\alpha_{kj}^{\delta}\mu_u(t)\delta(0)\bigr)e_j^{(m)}\phi_k(t),\end{aligned}\tag{48}$$

time-varying components $\alpha_{kj}(t)=\alpha_{kj}^{-}+\alpha_{kj}^{\delta}\mu_u(t)\delta(0)$ with $\alpha_{kj}(t^+)=\alpha_{kj}^{-}+\alpha_{kj}^{\delta}\mu_u(t)\delta(0)$ and $\alpha_{kj}(t^-)=\alpha_{kj}^{-}$. The (possibly discontinuous) components to the left and right of any $t\in\mathbf{R}_0^+$ represented by

$$u_j(t^{\mp})=\sum_{k=1}^{\infty}\sum_{j=1}^{n}\alpha_{kj}(t^{\mp})\phi_k(t).$$

That representation is well posed since $u,u',u''\in L_2^m$ so that both u' and u'' admit representations with real constant components α_{kj}^{-} and $\alpha_{kj}^{\delta}\delta(\tau-t)$ for input impulses at $\tau=t$. By convenience, the representations of operators for zero initial state and zero input are discussed separately.

5.1. Representations of the zero-state relevant operators

Note that the state trajectory solution satisfies

$$x(t)]_{\varphi\equiv 0}=\bigl(S^u u\bigr)(t)=\int_0^t H_x(t,\tau)u(\tau)\,d\tau=\int_0^t T(t,\tau)B(\tau)u(\tau)\,d\tau\tag{49}$$

with possible eventual discontinuities. Thus, the kernel $H_x(t,\tau)=T(t,\tau)B(\tau)$ of $S^u\in\mathbf{L}(U,X)$ admits the representation $H_x(t,\tau)=\sum_{k=1}^{\infty}\psi_{xk}(t)\theta_k^T(\tau)$, where

$$\begin{aligned}\psi_{xk}(t)&=\sum_{j=1}^{m}\int_0^t\alpha_{kj}(t)T(t,\tau)B(\tau)\bigl(e_j^{(m)}\phi_k(\tau)\bigr)\,d\tau\\&=\sum_{k=1}^{\infty}\int_0^t T(t,\tau)B(\tau)\underline{\alpha_k}(t)\phi_k(\tau)\,d\tau\end{aligned}\tag{50}$$

is the n-vector state for zero initial conditions with input

$$\sum_{j=1}^{m} \alpha_{kj}(t) e_j^{(m)} \phi_k(t) \text{ with } \underline{\alpha_k}(t) = \sum_{j=1}^{m} \alpha_{kj}(t) e_j^{(m)}$$

since

$$u_j(t) = \sum_{k=1}^{\infty} \alpha_{kj}(t)\phi_k(t) = \sum_{k=1}^{\infty} \underline{\alpha_k}(t)\phi_k(t),$$

and

$$u(t) = \sum_{\substack{k_j=1 \\ 1\leqslant j\leqslant m}}^{\infty} \underline{\alpha_{k}}_j(t)\big(e_j^{(m)}\phi_k(t)\big) = \sum_{k=1}^{\infty}\sum_{j=1}^{m} \alpha_{k_j}(t) e_j^{(m)} \phi_k(t), \tag{51}$$

where $\underline{\alpha_k}(t) = (\alpha_{k1}(t), \alpha_{k2}(t), \dots, \alpha_{km}(t))^T \in \mathbf{R}^m$. In the same way,

$$\begin{aligned} y(t) = \big(M^u u\big)(t) &= \int_0^t H(t,\tau)u(\tau)\,d\tau \\ &= \int_0^t \big(C(t)T(t,\tau)B(\tau) + D(t)\delta(\tau - t)\big)u(\tau)\,d\tau \end{aligned} \tag{52}$$

so that

$$\begin{aligned} y(t^-) &= \int_0^{t^-} H(t^-,\tau)u(\tau)\,d\tau = \int_0^{t^-} C(t)T(t^-,\tau)B(\tau)u(\tau)\,d\tau, \\ y(t^+) &= \int_0^{t^+} H(t^+,\tau)u(\tau)\,d\tau \\ &= \int_0^{t^+} C(t)T(t^+,\tau)B(\tau)u'(\tau)\,d\tau + D(t)\delta(0)\mu_u(t)u''(t) \end{aligned} \tag{53}$$

with the operator $H(\cdot,\cdot)$ admitting a representation:

$$H(t^-,\tau) = \sum_{k=1}^{\infty} \psi_k(t^-)\theta_k^T(\tau), \qquad H(t^+,\tau) = \sum_{k=1}^{\infty} \psi_k(t^+)\theta_k^T(\tau), \tag{54}$$

$$\psi_k(t^-) = \sum_{j=1}^{m} \int_0^{t^-} \alpha_{kj}(t^-)C(t)T(t^-,\tau)B(\tau)e_j^{(m)}\phi_k(\tau)\,d\tau,$$

$$\psi_k(t^+) = \sum_{j=1}^{m} \int_0^{t+} \alpha_{kj}(t^+)\big(C(t)T(t^+,\tau)B(\tau) + D(\tau)\delta(\tau - t)\mu_u(\tau)\big)e_j^{(m)}\phi_k(\tau)\,d\tau$$

$$= \sum_{j=1}^{m} \alpha_{kj}(t^+) \left(\int_0^{t+} C(t)T(t^+,\tau)B(\tau)\phi_k(\tau)\,d\tau + D(t)\delta(0)\mu_u(t)\phi_k(t) \right) e_j^{(m)}. \quad (55)$$

A quite similar representation may be obtained for the operator

$$T(t^{\mp},\tau) = \sum_{k=1}^{\infty} \psi_{Tk}(t^{\mp})\theta_k^T(\tau),$$

provided that $Bu \in L_2^n$, by replacing the real vector functions $\psi_k(t^{\mp})$ by vector functions $\psi_{Tk}(t^{\mp})$ obtained by fixing $C(t) = I_n$ in the right-hand sides of (55) and the use of the representation:

$$B(t)u(t) = \sum_{k=1}^{\infty}\sum_{j=1}^{m} \bar{\alpha}_{kj}(t)e_j^{(n)}\phi_k(t),$$

$$\psi_{Tk}(t^{\mp}) = \sum_{j=1}^{m} \int_0^{t^{\mp}} \bar{\alpha}_{kj}(t^{\mp})T(t^{\mp},\tau)e_j^{(n)}\phi_k(\tau)\,d\tau.$$

As a result,

$$x(t) = \sum_{k=1}^{\infty}\sum_{j=1}^{n} \beta_{xkj}(t)e_j^{(n)}\phi_k(t) = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\sum_{j=1}^{n} \alpha_{\ell j}(t)\psi_{Tk}(t)\int_0^t \theta_k^T(\tau)e_j^{(n)}\phi_\ell(\tau)\,d\tau,$$

$$y(t) = \sum_{k=1}^{\infty}\sum_{j=1}^{p} \beta_{kj}(t)e_j^{(p)}\phi_k(t) = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\sum_{j=1}^{n} \alpha_{\ell j}(t)C(t)\psi_{Tk}(t)\int_0^t \theta_k^T(\tau)e_j^{(n)}\phi_\ell(\tau)\,d\tau,$$

where $\beta_{xki}(t)$ and $\beta_{xk}(t)$ $(i = 1,2,\ldots,n;\ j = 1,2,\ldots,p;\ k = 0,1,\ldots)$ are the components of $x:[0,\infty) \to \mathbf{R}^n$ and $y:[0,\infty) \to \mathbf{R}^p$ with respect to the two respective orthonormal basis $\{e_j^{(n)}\phi_k\}_1^\infty$ and $\{e_j^{(p)}\phi_k\}_1^\infty$.

5.2. *Representations of the zero-input relevant operators*

Since the function of initial conditions $\varphi \in IC([-\bar{r},0),\mathbf{R}^n)$ has the form $\varphi(t) = \varphi_0(t) + (\tilde{\varphi}(t) + \tilde{\varphi}_{\mathrm{imp}}(t))$ with domain of finite measure, $\varphi_0(t)$ being uniformly bounded and $(\tilde{\varphi}(t) + \tilde{\varphi}_{\mathrm{imp}}(t))$ possessing a support of zero measure and a domain of finite measure. Then, φ, φ_0 and $(\tilde{\varphi}(t) + \tilde{\varphi}_{\mathrm{imp}}(t))$ are in L_2^m while $\varphi(t^+) \neq \varphi(t^-)$ if $\tilde{\varphi}(t) + \tilde{\varphi}_{\mathrm{imp}}(t) \neq 0$. Then it is possible to represent

$$\varphi(t) = \sum_{k=1}^{\infty}\sum_{j=1}^{n} \alpha_{\varphi kj}(t)e_j^{(n)}\phi_k(t)$$

with left and right limits:

$$\varphi(t^-) = \sum_{k=1}^{\infty}\sum_{j=1}^{n} \alpha_{\varphi kj}^- e_j^{(n)}\phi_k(t),$$

$$\tilde{\varphi}(t) + \tilde{\varphi}_{\mathrm{imp}}(t) = \sum_{k=1}^{\infty}\sum_{j=1}^{n} \left(\alpha_{\varphi kj}^+ - \alpha_{\varphi kj}^-\right)e_j^{(n)}\phi_k(t),$$

$$\begin{aligned}\varphi(t^+) &= \sum_{k=1}^{\infty}\sum_{j=1}^{n}\alpha_{\varphi kj}^{+} e_j^{(n)}\phi_k(t)\\ &= \sum_{k=1}^{\infty}\sum_{j=1}^{n}\alpha_{\varphi kj}^{-} e_j^{(n)}\phi_k(t) + \sum_{k=1}^{\infty}\sum_{j=1}^{n}\big(\alpha_{\varphi kj}^{+} - \alpha_{\varphi kj}^{-}\big) e_j^{(n)}\phi_k(t)\end{aligned}$$

with real components $\alpha_{\varphi kj}(t^-) = \alpha_{kj}^-$ and $\alpha_{\varphi kj}(t^+) = \alpha_{\varphi kj}^+ = \alpha_{kj}^- + \alpha_{kj}^{\delta}\mu_\varphi(t)\delta(0)$ in a similar way as argued for the input representation. The zero-input trajectory of S is given by

$$\begin{aligned}x(t) = (S\varphi)(t) = \sum_{k=1}^{\infty}\psi_{Tk}(t)\Bigg[\sum_{\ell=1}^{\infty}\sum_{j=1}^{n}\alpha_{\varphi\ell j}\Bigg(&\theta_k^T(0)e_j^{(n)}\phi_\ell(0)\\ &+ \int_{-\bar r}^{0}\theta_k^T(\tau+\bar r)e_j^{(n)}A_{\mathrm d}(\tau+\bar r)\phi_\ell\big(\tau+\bar r - r(\tau)\big)\Bigg)\Bigg]\end{aligned} \tag{56}$$

while its zero-input output state trajectory follows directly from the above expression by premultiplying its right-hand side by $C(t)$ to obtain a point-wise representation of the operator M via $y(t) = (M\varphi)(t) = C(t)x(t) = (C(t)S\varphi)(t)$.

5.3. Representations of the zero-state relevant operators associate with gate operators

If

$$u(t) = u''(t) = \sum_{t_i\in TU}\omega(t_i)\delta(t-t_i)u_0(t)$$

is impulsive with $u_0 \in L_2^m$ and $\omega : [0,\infty) \to \mathbf{R}$ being a piecewise continuous weighting function, the so-called gate-operators [15], are relevant what occurs in certain applications of electronics. Then, one has the point-wise operator definitions:

$$\big(S^u u\big)(t) = \int_0^t H_x(t,\tau)u(\tau)\,d\tau = \int_0^t \sum_{t_i\in TU} T(t,\tau)B(\tau)\omega(\tau)\delta(\tau-t_i)u_0(\tau)\,d\tau, \tag{57}$$

$$\big(S^{u_0}u_0\big)(t) = \sum_{t_i\in TU} T(t,t_i)B(t_i)\omega(t_i)u_0(t_i), \tag{58}$$

$$\begin{aligned}\big(M^u u\big)(t) = \int_0^t H(t,\tau)u(\tau)\,d\tau = \int_0^t \sum_{t_i\in TU} &C(t)T(t,\tau)B(\tau)\omega(\tau)\delta(\tau-t_i)u_0(\tau)\,d\tau\\ &+ D(t)\mu_u(t)\omega(t)\delta(0)u_0(t),\end{aligned} \tag{59}$$

$$\big(M^{u_0}u_0\big)(t) = \sum_{t_i\in TU} C(t)T(t,t_i)B(t_i)\omega(t_i)u_0(t_i) + D(t)\mu_u(t)\omega(t)\delta(0)u_0(t). \tag{60}$$

The following result holds.

Theorem 3. *The operator $(S^{u_0}u_0)(t)$, defined in* (58)*, is a Hilbert–Schmidt operator if $B:[0,\infty)\to \mathbf{R}^{m\times n}$ is in $L_\infty^{m\times n}$, $T(t,\tau)\in L_2^{n\times n}$ as a function of τ over $[0,\infty)$ for all $t\in \mathbf{R}_0^+$ and $\sum_{t_i\in TU}\omega^2(t_i)<\infty$ (which holds if $\omega\in L_2$).*

The operator $(M^{u_0}u_0)(t)$, defined in (60)*, is a Hilbert–Schmidt operator if $C:[0,\infty)\to \mathbf{R}^{n\times p}$ is in $L_\infty^{n\times p}$, $T(t,\tau)\in L_2^{n\times n}$ as a function of τ over $[0,\infty)$ for all $t\in \mathbf{R}_0^+$, $\sum_{t_i\in TU}\omega^2(t_i)<\infty$ and $D(t)\equiv 0$ for all $t\in \mathbf{R}_0^+$.*

Proof. $(S^{u_0}u_0)(t)$ and $(M^{u_0}u_0)(t)$ are of Hilbert–Schmidt if their kernels are of Hilbert–Schmidt (i.e. square-integrable on$(-\infty,\infty)$), [14,15], what holds under the given respective conditions. □

5.4. Compactness of the relevant input-state, state-output and input–output operators

The main well-known properties of compact operators, which are useful to approximate infinite-dimensional spaces by finite-dimensional ones, used are the following ones [14,15]: Hilbert–Schmidt operators and operator of finite-dimensional ranges are compact. A degenerate operator from a Banach's space to a Banach's space which is the limit of operators of finite-dimensional ranges is compact. Finally, an operator of closed range which is compact has a finite-dimensional range. From the developments of Sections 5.1–5.3, Theorem 3, and the above list of properties, the subsequent result holds for relevant operators defined in (10), (11) and some related operators.

Theorem 4. (i) *Operators $S\in \mathbf{L}(L_2^n,L_2^n)$, $M\in \mathbf{L}(L_2^n,L_2^p)$, $SS^*\in \mathbf{L}(L_2^n,L_2^n)$, $MM^*\in \mathbf{L}(L_2^n,L_2^n)$ are compact.*

(ii) *Let the input $u\in U\subset L_2^m$ be impulsive and generated from a mapping $U_0\times W\to U$ (gate operator) via a reference input $u_0\in U_0\subset L_2^m$ and $\omega\in W\subset L_2$ is a modulating weighting function that generates an impulsive input from u_0 defined by*

$$u(t)=\sum_{t_i\in TU}\omega(t)\delta(t-t_i)u_0(t)=\sum_{t_i\in TU}\omega(t_i)\delta(0)\omega(t_i)\mu_u(t).$$

Assume that

$$\int_0^\infty\int_0^\infty \left\|T(t,\tau)\right\|^2 d\tau\, dt<\infty,$$

or in particular that the unforced S *is GES* (*see Theorem* 2 *for sufficiency-type conditions*)*. Then, the operators S^{u_0}, $(S^{u_0})^*$ and $S^{u_0}(S^{u_0})^*$, from appropriate Banach's spaces to appropriate Banach's spaces, are compact if all the entries of $B(t)$ are bounded and $\sum_{t_i\in TU}\omega^2(t_i)<\infty$. If, in addition, all the outputs of $C(t)$ are uniformly bounded and $D(t)\equiv 0$ for all $t\in \mathbf{R}_0^+$ then the operators M^{u_0}, $(M^{u_0})^*$ and $M^{u_0}(M^{u_0})^*$ are Hilbert–Schmidt operators and then compact.*

(iii) *Assume that the input satisfies the constraints in* (ii)*. Then operator $S^{u_0(k)}$ has a decomposition $S^{u_0(k)}=\sum_{i=1}^n \chi_i\Delta_i$ where $(\Delta_i x)(t)=\langle x,\xi_i\rangle_{L_2^n}\xi_i(t)$ $(i=1,2,\dots,n)$ while*

$$T^{(k)}(t,\tau)=\sum_{i=1}^k \chi_i\xi_i(t)\xi_i^T(\tau)=\sum_{i=1}^k \chi_i\psi_i(t)\theta_i^T(\tau),$$

where $\xi_i(t)$ is a set of orthonormal eigenvectors of respective eigenvalues χ_i $(i = 1, 2, \dots, n)$ of $T^{(k)}$ satisfying

$$\sum_{i=1}^{\infty} |\chi_i|^2 < \infty.$$

Furthermore,

$$S^{u_0} - S^{u_0(k)} = \sum_{i=k+1}^{\infty} \chi_i \Delta_i$$

and, since $\|S^{u_0}u_0\|_{L_2^n} = \langle S^{u_0}u_0, S^{u_0}u_0\rangle_{L_2^n}$,

$$\|S^{u_0}u_0\|_{L_2^n} = \sum_{j=1}^{m}\sum_{k=1}^{\infty} |\alpha'_{kj}|^2 |\chi_k|^2 \leqslant \sup_{t \in S^{u_0}u_0} \left(\|B(t)\|_2^2\right)\left(\sum_{j=1}^{m}\sum_{k=1}^{\infty} |\alpha_{kj}|^2 |\chi_k|^2\right)$$

provided that

$$u_0(t) = \sum_{j=1}^{m}\sum_{k=1}^{\infty} \alpha_{kj} e_j^{(m)} \phi_k(t) \in L_2^m \quad \textit{and} \quad B(t)u_0(t) = \sum_{j=1}^{m}\sum_{k=1}^{\infty} \alpha'_{kj} e_j^{(m)} \phi_k(t) \in L_2^m$$

since $B(t)$ is bounded on $[0, \infty)$. If $\|u_0\| \in L_2^m$ then $\sum_{k=1}^{\infty} |\alpha_{kj}|^2 = 1$, any integer $j = 1, 2, \dots, m$; so that

$$\|S^{u_0}u_0\|_{L_2^n} \leqslant \sup_{t \in S^{u_0}u_0} \left(\|B(t)\|_2^2\right)\left(\sum_{j=1}^{m} |\chi_j|^2\right).$$

Proof. (i) Define the sequences of operators

$$\Psi_{A0}^{(k)}(t) := \sum_{\ell=0}^{k} \frac{A_0^\ell t^\ell}{\ell!} \quad \text{and}$$

$$T^{(k)}(t^-, 0) := \Psi_{A0}^{(k)}(t)\Bigg[I + \int_{0^-}^{t^-} \Psi_{A0}^{(k)}(\tau)\tilde{A}(\tau)T^{(k)}(\tau, 0)\,d\tau + \int_{\bar{r}}^{t^-} \Psi_{A0}^{(k)}(\tau)A_{\mathrm{d}}(\tau)T^{(k)}\big(\tau - r(\tau), 0\big)\mathbf{1}(t - \tau)\,d\tau\Bigg],$$

$$\forall t \in \mathbf{R}_0^+,\ k = 0, 1, \dots,$$

with $T^{(0)}(0,0) = T(0,0) = I$, and $T^{(k)}(t,\tau) = T(t,\tau) = 0$, $\forall \tau > t$. Assume that for $-\bar{r} \leqslant \tau < t$ and any given finite real $t > 0$,

$$\|T^{(k)} - T\|_{\mathbf{L}(L_2[0,t], L_2[0,t])} \to 0.$$

Since

$$\lim_{k\to\infty}\left(\|\Psi_{A0}^{(k)}(t) - \Psi_{A0}(t)\|_2\right) = 0 \quad \forall t \in \mathbf{R}_0^+,$$

it follows by using ℓ_2-matrix norms that

$$
\begin{aligned}
&\left\|T^{(k)}(t^-+\varepsilon,0)-T(t^-+\varepsilon,0)\right\|_2\\
&\quad\leqslant 2\left\|\sum_{j=k+1}^{\infty}\Psi_{A0}^{(k)}(t+\varepsilon)-\Psi_{A0}(t+\varepsilon)\right\|_2\\
&\qquad\times\left[\int_0^{t^-+\varepsilon}\left\|\sum_{j=k+1}^{\infty}\Psi_{A0}^{(k)}(\tau)-\Psi_{A0}(\tau)\right\|_2^2\left(\left\|\tilde{A}(\tau)\right\|_2+\left\|\tilde{A}_{\mathrm{d}}(\tau)\right\|_2\right)^2 d\tau\right]^{1/2}\\
&\qquad\times\left[\sup_{0\leqslant\tau'\leqslant\tau-r(\tau)}\left(\int_{0^-}^{t^-}\left\|T^{(k)}(\tau-\tau',0)-T(\tau-\tau',0)\right\|_2^2 d\tau\right)\right.\\
&\qquad\left.+\sup_{\tau-r(\tau)\leqslant\tau'\leqslant\tau}\left(\int_{t^-}^{t^-+\varepsilon}\left\|T^{(k)}(\tau-\tau',0)-T(\tau-\tau',0)\right\|_2^2 d\tau\right)\right]\to 0\quad\text{as } k\to\infty
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\|T^{(k)}(t^++\varepsilon,0)-T(t^++\varepsilon,0)\right\|_2\\
&\quad\leqslant\left[\sum_{j=k+1}^{\infty}\sum_{t_\ell\in TN(t,t+\varepsilon)\cup TN_d(t,t+\varepsilon)}^{\infty}\left\|\Psi_{A0}^{(k)}(t_\ell)-\Psi_{A0}(t_\ell)\right\|_2\left(\left\|\tilde{A}(t_\ell)\right\|_2+\left\|\tilde{A}_{\mathrm{d}}(t_\ell)\right\|_2\right)d\tau\right]\\
&\qquad\times\left[\sup_{t_\ell\in TN(0,t)\cup TN_d(0,t)}\ \sup_{0\leqslant\tau\leqslant t_\ell-r(t_\ell)}\left(\left\|T^{(k)}(t_\ell-\tau,0)-T(t_\ell-\tau,0)\right\|_2\right)\right.\\
&\qquad\left.+\sup_{t_\ell\in TN(0,t)\cup TN_d(0,t)}\ \sup_{t_\ell-r(t_\ell)\leqslant\tau\leqslant t_\ell}\left(\left\|T^{(k)}(t_\ell-\tau,0)-T(t_\ell-\tau,0)\right\|_2\right)\right]\to 0\\
&\qquad\text{as } k\to\infty
\end{aligned}
$$

since $\|\Psi_{A0}^{(k)}(t)-\Psi_{A0}(t)\|_2\to 0$ as $k\to\infty$, $\forall t\in\mathbf{R}_0^+$. Furthermore,

$$T^{(k)}(t,\tau)=\sum_{\ell=1}^{k}\psi_\ell^{(k)}(t)\theta_\ell^{(k)^T}(\tau)$$

is of finite range for all t, τ for all finite integer k. Since

$$\left\|\psi^{(k)}(t^+)\right\|_2\leqslant k_1^{(k)}\left\|\psi^{(k)}(t^-)\right\|_2\quad\text{then }\left\|\psi(t^+)\right\|_2\leqslant k_1\left\|\psi(t^-)\right\|_2$$

from Theorem 1(ii) for some nonnegative real constants $k_1^{(k)}$ $(k=0,1,\ldots)$ and k_1, then

$$
\begin{aligned}
&\left\|T^{(k)}(t^++\varepsilon,0)-T(t^++\varepsilon,0)\right\|_2\leqslant\left\|T^{(k)}(t^-+\varepsilon,0)-T(t^-+\varepsilon,0)\right\|_2\to 0\\
&\quad\text{as } k\to\infty,\ \forall t\in\mathbf{R}_0^+,
\end{aligned}
$$

and $\|T^{(k)}-T\|_{L_{2e}^n}\to 0$, i.e.

$$\int_{0^-}^{t}\left\|T^{(k)}(\tau,0)-T(\tau',0)\right\|d\tau<\infty,\quad\text{as } k\to\infty,\ \forall t\in\mathbf{R}_0^+$$

and $T^{(k)}(\tau,0)$ is of finite range on $[0,t+\varepsilon)$, provided it is compact on $[0,t)$ for any finite integer $k\in\mathbf{Z}_0^+$. Since $\|T^{(k)}(t,0)-T(t,0)\|_{L_2^n}\to 0$ and $T^{(k)}(t,0)$ is of finite-dimensional range as $k \to \infty$, $\forall t \in \mathbf{R}_0^+$, $T(t,0)$ is a compact operator. It is now proved that T^*, T^*T and TT^* are also compact. Note that taking norms in L_2^n:

$$\begin{aligned}\left\|T^*T(f_n - f_m)\right\|_{L_2^n}^2 &= \left\langle T^*T(f_n - f_m), T^*T(f_n - f_m)\right\rangle_{L_2^n} \\ &= \left\langle TT^*T(f_n - f_m), T(f_n - f_m)\right\rangle_{L_2^n} \\ &\leqslant \left\|T(T^*T)(f_n - f_m)\right\|_{L_2^n}\left\|T(f_n - f_m)\right\|_{L_2^n} \to 0\end{aligned}$$

as $\mathbf{Z}_0^+ \ni n, m \to \infty$ for any bounded sequence $\{f_n\}_1^\infty$ and its endowed norm from the inner product in the Hilbert space of square-integrable real vector functions of dimension n. Then T^*T is a compact operator and, therefore, TT^* is compact as well [14]. As a result, weak convergence (i.e. convergence of the inner products) implies strong convergence (i.e. convergence of the sequences) so that $f_k \to f$ as $k \to \infty$ implying $\langle T^*Tf_k, f_k\rangle_{L_2^n} \to 0$ as $k \to \infty$ implies that $T^*Tf_k \to 0$ as $k \to \infty$ and

$$\left\|T^*(f_n - f_m)\right\|_{L_2^n}^2 \leqslant \left\|TT*(f_n - f_m)\right\|_{L_2^n} \cdot \|f_n - f_m\|_{L_2^n} \leqslant 2C\left\|TT*(f_n - f_m)\right\|_{L_2^n},$$

since

$$\|f_n - f_m\|_{L_2^n} \leqslant 2C\left\|TT*(f_n - f_m)\right\|_{L_2^n} \to 0$$

for some finite positive real constant C as $\mathbf{Z}_0^+ \ni n, m \to \infty$. Then, $\{T^*f_n\}_1^\infty$ converges so that T^*, and then S, SS^*, M and MM^*, are compact so that the operators are also compact and (i) has been proved.

(ii) Since

$$u(t) = u''(t) = \sum_{t_i \in TU} \omega(t_i)\delta(t - t_i)u_0(t)$$

and $(S^{u_0}u_0)(t)$ and $(M^{u_0}u_0)(t)$ are defined via (58)–(60), it follows that their kernels are square-integrable if $\sum_{t_i \in TU} \omega^2(t_i) < \infty$ and S^{u_0} and M^{u_0} are Hilbert–Schmidt and then compact operators. In the same way as in (i), $S^{u_0}(S^{u_0})^*$, $M^{u_0}(M^{u_0})^*$, $(S^{u_0})^*$ and $(M^{u_0})^*$ are compact since S^{u_0} and M^{u_0} are compact.

(iii) It follows directly from the definitions of the various operators and their spectral decompositions since they are compact since the reference input is square-integrable. □

Under a similar reasoning as that used in Theorem 4(i), it may be proved that the operators $\Psi_{A0} \in \mathbf{L}(IC, L_2^n)$, $S_{A0\mathrm{e}} \in \mathbf{L}(IC, L_{2\mathrm{e}}^n)$, $M_{A0\mathrm{e}} \in \mathbf{L}(IC, L_{2\mathrm{e}}^n)$, $S_{A0} \in \mathbf{L}(IC, L_2^n)$ and $M_{A0} \in \mathbf{L}(IC, L_2^n)$ are compact. The evolution operator $\Psi_A \in \mathbf{L}(IC, L_2^n)$ as well as $S_{A\mathrm{e}} \in \mathbf{L}(IC, L_{2\mathrm{e}}^n)$, $M_{A\mathrm{e}} \in \mathbf{L}(IC, L_{2\mathrm{e}}^p)$, $S_A \in \mathbf{L}(IC, L_2^n)$ and $M_A \in \mathbf{L}(IC, L_2^p)$ are also compact under similar proving guidelines. Those properties are useful to approximate the zero-input responses of the auxiliary systems S1 and S2 through finite-dimensional real vector functions what holds irrespective of the stability or not of the infinitesimal generator A_0 or that of $A(t)$.

Remark. 1. Theorem 4(ii) also holds under similar conditions if $U_0 \in L_2^m$, $W \in L_2^{m\times m}$ and $\sum_{t_i \in TU} \|\omega(t_i)\|^2 < \infty$ what implies that $U \subset L_2^m$. The spectrum of T, $\mathrm{sp}(T)$, is close, belongs to $[-\|T\|, \|T\|]$ and includes zero. All the points of $\mathrm{sp}(T)/\{0\}$ are isolated and eigenvalues of T. Such a spectrum is either zero, finite (when excluding zero) or a sequence which converges to zero [14].

2. $\Psi_{A0}(t) = \sum_{k=0}^{\bar{\mu}-1} \alpha_k(t) A_0^k$ for any integer $\bar{\mu} \geqslant \mu$, μ being the degree of the minimal polynomial of A_0 and $\alpha_k : \mathbf{R}_0^+ \to \mathbf{R}$ $(k = 0, 1, \ldots, \bar{\mu} - 1)$ are linearly independent functions calculated from an algebraic system of linear equations [1]. As a result, each unique real n-dimensional vector trajectory solution of S1 is represented in the form

$$z_{A0}(t) = \sum_{k=0}^{\bar{\mu}-1} \alpha_k(t) \left(\sum_{i=1}^{n} c_i A_0^k e_i^{(n)} \right)$$

for any initial condition

$$x_0 = \sum_{i=1}^{n} c_i e_i^{(n)}, \quad c_i \in \mathbf{R},\ i = 1, 2, \ldots, n,$$

so that it is of finite dimension as it is the real p-vector function $C(t) z_{A0}(t)$. As a result, the operators $S_{A0} \in \mathbf{L}(IC, L_2^n)$, $S_{A0\mathrm{e}} \in \mathbf{L}(IC, L_{2\mathrm{e}}^n)$, $M_{A0} \in \mathbf{L}(IC, L_2^p)$, $M_{A0\mathrm{e}} \in \mathbf{L}(IC, L_{2\mathrm{e}}^p)$. If, furthermore, $\|A_{\mathrm{d}}\| \in L_2$ then $\bar{S}_{A0\mathrm{e}} \in \mathbf{L}(L_{2\mathrm{e}}^n, L_{2\mathrm{e}}^n)$, $\bar{M}_{A0\mathrm{e}} \in \mathbf{L}(L_{2\mathrm{e}}^n, L_{2\mathrm{e}}^p)$, $\bar{S}_{A\mathrm{e}} \in \mathbf{L}(L_{2\mathrm{e}}^n, L_{2\mathrm{e}}^n)$ and $\bar{M}_{A\mathrm{e}} \in \mathbf{L}(L_{2\mathrm{e}}^n, L_{2\mathrm{e}}^p)$ are also compact [see (4c), (4d), (4g) and (4h) in Theorem 1].

3. If $\varphi_{\mathrm{imp}} \equiv 0$ for $t \in [-\bar{r}, 0]$ then $S_{\mathrm{e}} \in \mathbf{L}(IC, L_{2\mathrm{e}}^n)$ and $M_{\mathrm{e}} \in \mathbf{L}(IC, L_{2\mathrm{e}}^p)$ are compact if the auxiliary system S3, or equivalently the unforced S, equations (1), is GAS.

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